LOCALIZATION OF WAVES

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Protect me from knowing what I don't need to know
Protect me from even knowing that there are things to know that I don't know
Protect me from knowing that I decided not to know about
the things that I decided not to know about
Douglas Adams, 1992 in: *Mostly Harmless*

1. Introduction

Writing a short and critical review about localization of waves is an ambiguous project, doomed to be incomplete, and doomed to be unfair to the many good papers that have been written on this fascinating subject. At least every subject raised in this modest contribution merits, I think, an entire review and, fortunately, some exist already. By no means is this review going to be a replacement for the excellent works by Thouless [1], Ramakrishnan [2], Mott [3], Souillard [4], John [5, 6], and Vollhardt and Wölfle [7]. Answering the question whether or not reported experiments reveal wave localization is not the aim of this review either. The emphasis in this review will be on definitions, features and consequences.

The phenomenon of localization appears in many disguises, sometimes an "unrecognizable monster" according to its creator Anderson in looking back at 25 years of localization [8]. Originally, the concept of localization came forward out of the wish to understand metal-insulator transitions [3], where "disorder" prevents the electrons in a semiconductor to move freely. Like all great discoveries it initiated a new way of thinking, with impact in domains far from the one it originated, like radiative transfer [5, 9], seismology [10], and even atomic physics [11] and high-energy physics [12]. Localization has been at the origin of a new "mesoscopic" physics, i.e. a physics between microscopic and macroscopic, with best of both worlds, namely disorder and phase. Finding convincing evidence for localization has been a constant challenge for experimental physicists working with waves in disordered media. Only a handful of experiments claim observation of localization in a very direct way. Nevertheless, many new phenomena - coherent backscattering, weak localization, long-range correlations, resonant multiple scattering to mention a few - have been discovered on the way. In all reports of localization, "alternative" explanations are possible. This has largely contributed to the actual confused state that localization seems to be in, at least for the younger generation. Several lectures given in Les Houches in march 1998, two generations after the discovery, made this very clear.

Most theoretical approaches are extremely intuitive, especially the good ones. A good example is the scaling theory of localization [13], inspired by the successful renormalization theory of statistical physics. More technical methods, using Green's functions, diagrams, spontaneous symmetry breaking and all those things that many always wanted to avoid, have been guided by a good deal of physical intuition, ranging from the first "locator expansion" by Anderson [14], to the microscopic "diagrammatic" theories [15, 16] developed in the end of the seventies, the "non linear sigma" models [17, 18, 19] borrowed from quantum field theory. They all aim to find and verify "great principles" and "simple ideas" in a mess of complicated physics. In the eighties, rigorous mathematical proofs [20] started to appear that showed that localization really exists in the regime where physical intuition believed it to occur, briefly that Anderson was right after all. As a result, they did not shock the physics community that much. Yet to me, mathematical theories showed how subtle the concept is when trying to make really strong statements, a subtlety that is not evident from theoretical treatments.

Two models have learned us a lot about Anderson localization. Both are idealizations of the real world, but contain the physics that is believed to be essential. The most recent one is

random matrix theory. This theory is now in rapid development and applies to open "quasi one dimensional" random systems [21], an elegant theory where symmetry plays a dominant role, the rest "just" being calculation. I found it funny to learn that this theory actually assumes a disordered medium to be a stack of Sinai billiards, to each of which one applies chaos theory developed by Dyson. I suppose an experimentalist would never have thought of his sample this way. The second model is the fundamental Anderson Hamiltonian, which models the hopping of a tightly bound electron from one site to the other, either in an infinite lattice or in a finite closed lattice. Many thorough numerical studies have been carried out with this Hamiltonian.

Giving up any of these simplifications rapidly makes the problem unsolvable, leaving room for all kinds of wild speculations. Altogether, the subject is hard and perhaps too vast to comprehend for an interested newcomer in the field, not knowing what treatment is best suited to study her or his particular problem, but eager to understand the impact of localization in her or his field.

2. What is Localization?

The phenomenon of localization can occur when waves propagate in some kind of random medium. However, the exact definition of localization does not seem to be unique. Perhaps it is easier to tell what localization is certainly not. Localization is not the same thing as "weak localization" or "coherent backscattering". These mesoscopic phenomena refer to interference effects in multiple scattering that do even persist for low disorder. Localization is not trapping either, which would occur by simply surrounding the wave with a bunch of ideal mirrors, an effect that even persists classically. Equivalently, a state in the gap of a semi-conductor or a photonic band gap material, created by some local defect, is not called "localization" in the sense of the present paper. It is just a bound state. As I will point out below, localization requires a *finite number of bound states per unit volume*. This definition is strong in infinite media, but ambiguous in a finite medium since even one bound state has finite density in a finite volume. In an open system, the spectrum is continuous and microstates cannot be counted. Localization in open systems is often associated with an exponentially small ensemble-averaged transmission, which is - I think - only one aspect of localization. Fluctuations are at least as important.

2.1. LOCALIZATION IN INFINITE SYSTEMS

The easiest definition is the one suggested by the title of the original paper by Anderson, namely the "absence of diffusion". Many authors have accepted the criterion

$$D(E) = 0 \,, \tag{1}$$

as a working criterion for localization for waves at energy E. Indeed, the diffusion constant $D(E)$ at energy E seems to be a relevant quantity because it is strongly related to ensemble-averaged conductance, and that is a basic quantity measured in the laboratory, at least for conductors at that time. I want to stress here immediately that the diffusion constant is typically a transport quantity in an *infinite* system, whereas "conductance" is a property of an *open finite* system. We now know that this distinction is crucial in any discussion on localization.

Some people go even one step further by defining localization as a "small" diffusion constant, e.g. by comparing to the diffusion coefficient anticipated from classical transport. This puts the phenomenon on the same footing as "weak localization". The latter is sometimes seen as a precursor of strong localization, but the exact relation is far from evident. Small

optical diffusion constants have been reported [22]. An interpretation in terms of localization is dangerous as reasons other than localization exist why diffusion can be suppressed [23].

The restrictions of the criterion (1) can be revealed by recalling the Einstein random walk formula for the diffusion constant,

$$D = \frac{1}{2d} \lim_{t \to \infty} \left\langle \frac{\mathbf{r}^2}{t} \right\rangle, \qquad (2)$$

where $\langle \cdots \rangle$ stands for ensemble averaging over disorder, d is the dimension, $\mathbf{r}$ is the center-of-mass of the wave packet in space and t denotes time. Apparently, criterion (1) imposes that $\mathbf{r}^2 < t$ for an ensemble-averaged medium, but that does definitely not "localize" the wave packet for a typical realization of the disorder. To have that we must impose the much stronger constraint that

$$\mathbf{r}^2 < \text{constant}, \qquad \text{almost surely}. \qquad (3)$$

Mathematicians call this dynamical localization. I would like to warn the reader that this terminology is not equal to what a physicist usually calls dynamical localization (section 3.1.1)

A definition for localization exists that circumvents time evolution and ensemble averaging from the start. Localization in a certain energy interval ΔE is said to occur when, *for "almost every realization"* $\{V\}$ of the random potential V, the random Hamiltonian $H = H_0 + V$ has, in the energy range ΔE only point spectrum with (at least) exponentially localized eigenfunctions $\psi_n(\mathbf{r})$, i.e.

$$|\psi_n(\mathbf{r})| \leq C_n(\{V\}) \exp\left(-A|\mathbf{r} - \mathbf{r}_n(\{V\})|\right) \quad \text{almost surely}. \qquad (4)$$

Here A is independent of the realization and identified with the maximal localization length in the interval ΔE. The "almost surely" requirement is very strong, but not too strong to be impossible. Because the eigenfunctions decay exponentially, the spectrum of the Hamiltonian in the interval ΔE will be "pure point". The difference with normal bound states is that the interval ΔE contains an *infinite* number of eigenvalues. The spectrum is said to be "dense pure point". In an infinite system, bound states with infinitely close energies are possible because they correspond to localized eigenstates that are infinitely far apart and hardly overlap. It is well known from e.g. quantum tunneling problems that any overlap of wavefunctions would cause eigenvalue repulsion. We will make this important notion more precise in the following section.

One consequence of the dense point spectrum is that the localized regime has, just like the conventional extended regime, a finite number of eigenstates $N(E, \Delta E)$ *per unit volume* in some finite energy interval ΔE, a number that can actually shown to be equal for "almost any" realization of the disorder [25]. This is rather unfortunate: The density of states $\rho(E) \equiv \lim_{\Delta E \to 0} N(E, \Delta E)/\Delta E$ is an important quantity in transport theory and is also experimentally accessible, but is - in an infinite system - not sensitive to localization. The absence of localization effects in the density of states implies that no signatures can be found in thermodynamic quantities such as the specific heat.

It is constructive to investigate the relation of definition (4) to the more dynamical one involving the time dependence of $\mathbf{r}(t)$. It is known that criterion (4) implies that (almost surely) $|\mathbf{r}(t)|/t = 0$ for large times t. Consequently, criterion (4) is stronger than the one in Eq. (1), but still not strong enough to obey criterion (3). It is also known that if $\mathbf{r}^2(t)$ is bounded than criterion (4) must be obeyed. The reverse is not true: Exotic models exist, that obey criterion (4) but have un unbounded $\mathbf{r}^2(t)$, and are thus not localized in the strict sense. To guarantee this property too, one must impose that $C_n(\{V\}) \leq C_\epsilon(\{V\}) \exp(\epsilon|\mathbf{r}_n(\{V\}|)$ for any small positive ϵ

[26], i.e. the localization length of states far apart must be "more or less" equal for almost any realization.

A common way to study localization is by means of the so-called *return probability* $P(\mathbf{r})$, defined as,

$$P(\mathbf{r}) = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \, |\langle \mathbf{r}| \exp(-iHt) \, |\mathbf{r}\rangle|^2 \ . \tag{5}$$

It gives the probability density - or when integrated over some volume a probability - for a particle leaving at position $\mathbf{r}$ to finally come back at the same point. We expect this quantity to vanish for an extended state, and to be nonzero for a localized state. By using the spectral decomposition,

$$\exp(-iHt) = \sum_n \exp(-iE_n t) \, |\psi_n\rangle \, \langle\psi_n| \tag{6}$$

it is not difficult to see that

$$P(\mathbf{r}) = \sum_n |\psi_n(\mathbf{r})|^4 \ . \tag{7}$$

This brings the return probability on the same footing as the *inverse participation number* $\mathcal{P}^{-1}(E)$ for an eigenstate at energy E, defined as

$$\mathcal{P}^{-1}(E) = \sum_i |\psi_E(\mathbf{r}_i)|^4 \ , \tag{8}$$

where the summation runs over all sites, or in the case of a continuum, involves an integral over space. The participation number measures the number of sites covered by the wave function at energy E, or equivalently the size of a state at energy E. The relation between participation ratio and the return probability evidently is $\sum_i P(\mathbf{r}_i) = \sum_n \mathcal{P}^{-1}(E_n)$. For an exponentially localized state with localization length ξ we estimate $\mathcal{P} \sim (\xi/a)^d$, with a the lattice spacing.

Another convenient way to study localization is by means of the spatial correlation functions. The localization length $\xi(E)$ can be defined from the intensity-intensity correlation function

$$\left\langle |\psi_E(\mathbf{r})|^2 |\psi_E(\mathbf{r}')|^2 \right\rangle \sim \exp\left(-\frac{|\mathbf{r} - \mathbf{r}'|}{\xi(E)} \right) \ . \tag{9}$$

I want to stress here that the field-field correlation function also decays exponentially, and defines the scattering mean free path or more exactly the extinction length $\ell(E)$

$$\langle \psi_E(\mathbf{r})\psi_E^*(\mathbf{r}') \rangle \sim \exp\left(-\frac{|\mathbf{r} - \mathbf{r}'|}{\ell(E)} \right) \ . \tag{10}$$

The ensemble-averaging is crucial here, because that's the mechanism responsible for the exponential decay. According to criterion (4) for the localized regime, exponential decay occurs for "almost any realization", and not only for the ensemble-average of the intensity correlation. Contrary to the field correlation function, the intensity correlation in Eq. (9) would decay algebraically in the extended regime.

2.2. CLOSED SYSTEMS

By definition, a closed system has boundary conditions $\psi = 0$ or $\partial_n \psi = 0$ on its boundaries. In both cases the current density, whose component normal to the surface is given by $J_n(\mathbf{r}) \sim$
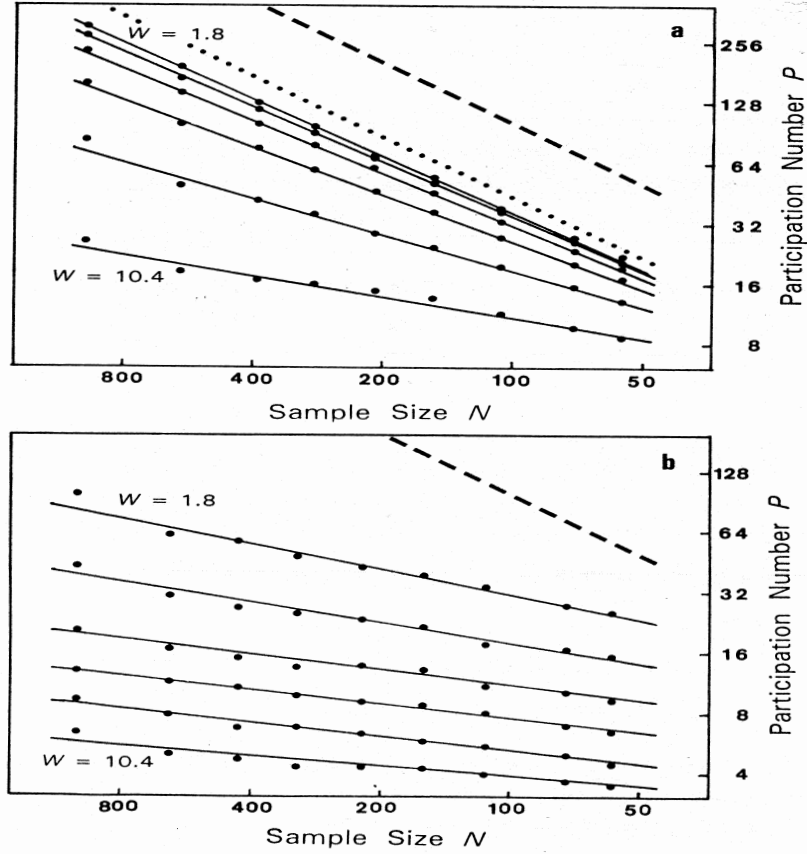
Figure 1. Participation ratio $\mathcal{P}(E)$ as a function of sample size N for the 2D Anderson model on a cubic lattice, for random energies $W/V = 1.8$ to 10.4. The top figure corresponds to an energy interval in the band center, the bottom figure to an energy interval just beyond the band edge. The broken line has $\mathcal{P}(E) = N$, the maximally possible value obtained for an extended Bloch plane wave. Taken from Schreiber [27], with kind permission from the author.

Im $\psi^*(\mathbf{r})\partial_n(\mathbf{r})\psi$, vanishes. Periodic boundary conditions are also encountered, transforming a finite system effectively into a periodic infinite one, from which no current can escape.

In a closed system all states are genuine bound states with *finite* level spacing , and the question is how this evolves as the system becomes bigger. I emphasize that numerically exact "*ab initio*" studies can only deal with finite-size systems, either closed or open.

We note that the inverse participation ration (8) is well defined in a closed system. In Figure 1 we show the participation ratio as calculated by Schreiber [27] using a careful numerical technique to solve the Anderson model on a 2D square lattice with N sites. What is important to note here is that the participation ratio declines as the disorder W increases. This is consistent with the fact that eigenstates in 2D are expected to be localized with a localization length ξ that shortens as the disorder W increases. Secondly, the participation ratio seems to be proportional to some power law of the number of sites, or equivalently to some power law of the system size

$$L \sim N^{1/d},$$

$$\mathcal{P}(E) \sim L^{\beta}. \tag{11}$$

where $0 \leq \beta \leq d$. This relation suggests that the eigenstates sort of interconnect in the sample according to some fractal structure, familiar from percolation studies. If the states are completely extended, one anticipates $\beta = d$. One the other hand, for exponentially localized states we do not expect that increasing the system size beyond the localization length will change a lot: $\beta = 0$. Finally, for localized eigenfunctions that decay algebraically $|\phi_E()| \sim 1/r^{\alpha}$ the tails sufficiently overlap to yield a finite β and one finds $\beta = 2d - 4\alpha$ [28, 29]. A nonzero fractal dimension may thus indicate algebraic decay of the eigenfunctions over a long range. The study of the fractal dimension β can therefore be very efficient to quantify localization in a *finite* system. We emphasize however that, contrary to classical percolation problems, it is ambiguous to define a critical disorder from these percolation arguments, since even exponentially localized states have overlapping tails due to quantum tunneling, and the connectivity depends on the "grey" scale used to display the percolation backbone.

A quantity that regularly shows up in recent considerations of Anderson localization in closed systems, in particular in relation with Dyson's random matrix theory of chaotic systems, is the level spacing distribution $P(s)$. It gives the probability density of finding two eigenvalues E_n with energy difference s. In the localized regime the exponential tails of distant states are hardly aware of each other. As a result, elementary statistical arguments suggest that the level spacing distribution is "Poisson",

$$P(s) = \frac{1}{\Delta} \exp\left(-\frac{s}{\Delta}\right) \quad \text{(localized regime)} \tag{12}$$

where Δ is the average level spacing and finite as long as a finite system is considered. This formula confirms that infinitely close levels are possible. We remark that Eq. (12) has assumed that the localization length of the levels ξ is much smaller than the sample dimension L. If not, we would not really speak of the localized regime, and a finite level spacing $\Delta E \approx 1/N(E) \sim L^{-d}$ would persist.

What can be expected in the extended regime? In principle all states overlap in space and one expects Eq. (12) to break down. It is tempting to apply ideas developed in chaos theory to this situation. The basic hypothesis in random matrix theory is that the energy E is the only constant of motion during the multiple scatterings and all other variables are completely scrambled. The only severe constrain is energy and the presence of time-reversal symmetry (for particles with spin there is one more). Time-reversal symmetry forces the matrix -element $\mathcal{H}_{nm} = \langle \psi_n | H | \psi_m \rangle$ to be real-valued. The rest is just a matter of evaluating the proper volume element (Jacobian) for the quantity to be considered. Time-reversal symmetry is characterized by a level spacing distribution,

$$P(s) = \frac{\pi s}{2\Delta^2} \exp\left(-\frac{\pi s^2}{4\Delta^2}\right) \quad \text{(extended regime; time reversal)} \tag{13}$$

whereas the absence of time-reversal symmetry is characterized by [30],

$$P(s) = \frac{32 s^2}{\pi \Delta^3} \exp\left(-\frac{4 s^2}{\pi \Delta^2}\right) \quad \text{(extended regime; no time reversal)} \tag{14}$$

Expressions (13) and (14) differ considerably from the Poisson distribution expected in the localized regime. In particular, the extended regime is characterized by $P(0) = 0$, a phenomenon called *level repulsion*, and by Eq. (12) absent in the localized regime.

The level spacing distribution functions have been thoroughly investigated for simplified models such as the Anderson model [83, 84]. Some results will be discussed in the next section. Equations (13) and (14), and in particular their behavior as $s \to 0$ are often taken as a definition of "quantum chaos", since the classical definition of "sensibility to initial conditions" turns out to be too strong for quantum systems. On the basis of their level spacing distribution, closed disordered systems can be called "chaotic" only in the extended regime.

2.3. OPEN SYSTEMS: TRANSPORT OF WAVES

The characteristics of localization discussed above assume that the system is either infinite or finite but closed. In 3D, both are not very realistic experimentally. After all, to see 3D localization one has to open somehow the system and relate the observed leaks to localization. Open systems are characterized by a nonvanishing current density on their boundary. The study of localization in open systems is a study of wave transport and transmission matrix of the system, and not one of eigenfunctions, which become continuum eigenfunctions. In an open system, the eigenstates of the closed system achieve a finite width, due to leakages through the boundaries. As first realized by Thouless this rather innocent notion enables to formulate a criterion for localization in open systems, which is by many - including myself - regarded as the most important result in localization theory after its discovery. This criterion will be discussed in Chapter 4.

In the following I will first discuss localization of electrons, considering originally only ensemble-averaged transport. Next, I will outline what random matrix theory has to say about localization. This theory focuses on fluctuations and probability distributions, setting a new trend for modern mesoscopic physics. These studies have seen experimental applications in microwave tubes and quantum dots. Finally, I discuss a promising numerical method that actually studies the resonant remnants of the localized states in open media. Especially in 2D, experiments directly probing the wave function are possible. Experimental studies of 2D localized wave functions of microwaves [31], acoustic waves [32] and bending waves [33] have been reported.

2.3.1. *Metal-Insulator Transition*

In electronic systems (metals) the whole concept of localization was devised to explain metal-insulator transitions. "Classical" transport theory predicts a disordered electronic system to be conducting. The word classical should - of course - be between quotes, because deep quantum-mechanical wave concepts such as "Bloch states" and "incompletely filled bands" have been necessary to understand why only deviations from perfect periodicity induce a nonzero electronic conductivity. Such impurities (phonons, vacancies or thermal fluctuations,) had been given a classical treatment.

A conceptual difference seems to exist between localization in *uncompensated* and *compensated* doped semi-conductors. Compensated means that the number of acceptor (donor) levels exceeds the number of valence electrons (holes). In uncompensated semi-conductors electronic transport is governed by "free" electrons colliding with impurities, having much in common with classical wave localization. In compensated semi-conductors, conduction is largely determined by impurity band propagation, i.e. quantum tunneling due to overlapping "tightly bound" quantum states of impurity levels. This mimics the physics of the Anderson model discussed later, although one

should not underestimate the role of electron-electron interactions and spin scattering that have not been taken into account by this model. All aspects play an important role in the solution of the critical exponents puzzle: Uncompensated materials, of which Si:P is the most famous, seem to have different critical exponents than compensated materials [34]. Recently, Shmilak etal. [35] suggested that the critical exponent of 1/2 is actually due to an incorrect extrapolation of the electronic conductivities towards zero temperature, and that all materials have a critical exponent close to one.

Let me summarize in a nutshell what Anderson localization implies for electron conduction. I stress once more that answering the important question whether or not experiments reveal localization is not the aim of this review.

The important transport coefficients, not only the electronic conductivity σ, but also thermal conductivity K and thermoelectric power S depend on temperature T and electron concentration n_e, and one would like to understand how. They can all be expressed in terms of the kinetic coefficients [36],

$$L(T, n_e) = \int_{-\infty}^{\infty} dE \, \frac{\partial f}{\partial E} D(E) \mathcal{L}[E - \mu(T)] \,. \tag{15}$$

Here $f(E, T)$ is the Fermi function. At finite temperature, transport coefficients always involve a finite energy width $\delta E \approx k_B T$, a crucial difference with classical wave transport. The generally accepted picture says that in the conducting regime, the diffusion constant vanishes according to a power law $D(E) \sim (E - E_c)^s$ with s some critical exponent, and E_c the mobility edge. Unfortunately, there is no consensus on the values for s. Different variants of the Anderson model (with and without broken time reversal, with and without spin, diagonal and off-diagonal spin interactions) exist that have different critical exponents. Nevertheless, the existence of critical exponents implies the transition to be continuous, as also predicted by the scaling theory of localization published in 1980, and not first-order, despite arguments put forward by Mott (see Chapter 4) in favor of the latter. The paper by Mott in 1972 [37] gives his point of view on the existence of a mimimum conductivity given the localization arguments put forward by Anderson and Thouless. In 3D he finds the rather small value $\sigma_{\min} \approx 0.025 e^2/\hbar a$. In reading the paper by Kaveh and Mott [38], published 10 years later and two years after the scaling theory had appeared, I got the impression that Mott never really gave up his minimum-conductivity version of localization.

As $T \to 0$, the Fermi level E_F is inversely proportional to the electron density and one sees that

$$\sigma(T = 0, n_e) \sim (E_F - E_c)^s \sim (n_e - n_c)^s \,. \tag{16}$$

This equation is, of course, the basis of electron localization. At the critical electron density one finds that $\sigma(T, n_c) \sim T^s$ and $K(T, n_c) \sim T^{s+1}$ [36]. Not all transport coefficients show critical behavior: at the critical density n_c the thermoelectric power reaches a large but finite value that is independent of temperature [36]. The complication in electronic systems is the presence of a second mechanism for an electronic phase transition, called Mott-Hubbard localization, and driven by electron-electron interactions. In addition, other mechanisms (phonons) may influence the thermal conductivity and thermopower.

In the insulating phase the dielectric constant $\varepsilon(E) \sim (E_c - E)^{-\nu}$ has the same critical exponent ν as the localization length $\xi(E)$ near the mobility edge E_c. At finite temperatures, conductance is still possible by thermally excited hopping from one localized state to the other. Mott showed that this "variable range hopping" leads to a temperature dependence [39]

$$\sigma(T) \sim \exp(-T_0/T)^{1/(d+1)} \tag{17}$$

in d dimensions. This law is widely observed [39].

Perhaps the most beautiful and direct manifestation of localized states in electronic 2D disordered media is the integer quantum Hall effect, observed first in 1980 [40]. Localization theory is not only capable to predict the plateau values of the Hall conductivity very precisely, it also explains the scaling in between, where the eigenstates go from localized to extended, as described by the scaling theory of localization [41, 42, 43].

2.3.2. *Mesoscopic Physics*

The study of classical wave localization is interesting because of the relatively large wavelengths involved. For classical waves it is much easier to measure the transmission coefficient from one angle (channel) to another (see Figure 2), as well as frequency and channel correlations. For a finite tube of length L with N transverse modes, classical diffusion theory tells that the ensemble-averaged transmission coefficient $T_{ab}(L)$ equals,

$$\langle T_{ab}(L) \rangle \sim \frac{\ell^*}{NL}, \tag{18}$$

where the transport mean free path ℓ^* is closely related to the diffusion constant D according to $D = v_E \ell^*/d$. Keeping definition (1) in mind, a "small" transmission may be the first signature of localization, exactly like in one dimension. In the localized regime, one may expect that

$$\langle T_{ab}(L) \rangle \sim \exp(-L/\xi), \tag{19}$$

with ξ the localization length. Such exponential decay has been reported both for microwaves [44] and light [45]. Important is, of course, to exclude absorption, which trivially leads to an exponential decay of the kind (19). In an open finite system, one can define localization as,

$$\xi(E) < L \qquad \text{(localization in open system)}. \tag{20}$$

We have seen earlier that - at least for an infinite medium - strict localization, in the sense of Eq. (4), is a property "for almost all" realizations of the disorder, and not only of a "typical" one. Of course, we would like to understand how this notion translates to the transmission coefficients. To this end it is constructive to consider not only ensemble-averages, as in Eq. (19), but the whole probability distribution. Fortunately, microwaves and acoustic waves facilitate such measurements. Except for the channel transmission coefficient T_{ab} one can consider the total transmission coefficient for a mode a,

$$T_a = \sum_b T_{ab}, \tag{21}$$

and the (dimensionless) conductance,

$$g \equiv \sum_{ab} T_{ab}. \tag{22}$$

This quantity makes the link with the conduction problem studied in electronic systems, since by the Landauer formula [21], g equals the conductance of a mesoscopic disordered system, except for a trivial factor $2e^2/h$ with the dimension of conductance. The presence of nonlocal
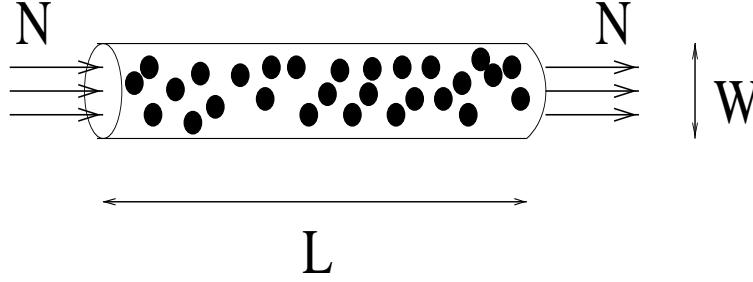
Figure 2. Geometry of a quasi one-dimensional system. The transverse size W is comparable to one mean free path; The length L is arbitrary. The tube has N input channels and N output channels.

mode correlations makes the total transmission and the conductance already nontrivial in the extended regime.

In closed systems, the energy eigenvalue distribution was seen to be a very characteristic feature. In open systems, the N transmission eigenvalues T_n of the transmission matrix T_{ab} take over this role. The transmission eigenvalue density $\rho(T)$ gives the probability density to find the eigenvalue T among the N transmission channels. Random matrix theory predicts $\rho(T)$ to be significantly different in extended and localized regime, as discussed below.

So far only random matrix theory established theoretical predictions for the probability distributions and correlation functions of channel transmission, total transmission and conductance in the localized regime, as well as in the transition regime. The strong point of random matrix theory is that it is nonperturbative, enabling to study all the way from the metallic (extended) regime to the strongly localized regime. At the time of writing, random matrix theory still has two severe limitations. First, it "only" applies to quasi-one dimensional systems, i.e. systems whose transverse width is at most comparable to the transport mean free path. The generalization to genuine 2D or 3D systems does not seem easy. Secondly, random matrix theory still is a stationary theory and is not yet capable to study spectral correlations or time tails. This last aspect is certainly going to be solved in the very near future.

2.3.3. *Statistics of Transmission in Extended Regime*

Elementary arguments show that, in the extended regime [46]

$$P(T_{ab}) = \frac{1}{\langle T_{ab} \rangle} \exp\left(-\frac{T_{ab}}{\langle T_{ab} \rangle}\right) \quad \text{(extended regime)} \tag{23}$$

i.e. the celebrated Rayleigh statistics for speckle intensities. On the basis of the law of large numbers one would expect the total transmission T_a to be Gaussian distributed with mean $\langle T_a \rangle \sim \ell^*/L$ and variance $(\Delta T_a)^2 = N \langle T_{ab} \rangle^2 \sim (\ell^*)^2/NL^2$. However, studies in the eighties have established that substantial nonlocal correlations exist between different channels [47]. This leads to the relatively large variance

$$(\Delta T_a)^2 \approx \frac{\ell^*}{NL} , \tag{24}$$

for the total transmission coefficient in the extended regime, i.e. a factor L/ℓ^* larger than what would have been expected if the N channels were independent. The long-range correlations also give rise to a nongaussian distribution for T_a [48].

The ensemble average $\langle g \rangle \sim N\ell^*/L$ can be related to the Ohmic conductance of a copper wire of length L and width A. This can be seen by realizing that $N \approx Ak^{d-1}$ in terms of the cross-section A of the wire and the (Fermi) wave number. The quantity $\ell^* k^{d-1}$ is nothing more than the Drude expression for the electronic conductivity, except again for a factor $2e^2/h$, containing only fundamental constants, and with the dimension of conductance. Again, due to long range correlations, fluctuations in g are much bigger than the naive result $\Delta g \sim \ell^*/L$ expected for N independent channels. The result [49],

$$\Delta g \approx 1 \qquad (25)$$

is known as a Universal Conductance Fluctuation, because it hardly depends on details of the system such as dimension, size or mean free path. Random matrix theory gives the exact result $\Delta g = \sqrt{2/15\beta}$, where $\beta = 1$ for time reversal symmetry, and $\beta = 2$ for broken time reversal symmetry [21]. Thus, broken time reversal symmetry kills the universal conductance fluctuations by a factor of *exactly* two. The factor $\Delta g = \sqrt{2/15}$ is confirmed by diagrammatic calculations for quasi 1D systems. For cubic 3D systems the somewhat larger factor $\sqrt{0.296}$ is found [50].

Random matrix theory predicts a fundamental relation between the probability distributions of transmission T_a and one-channel transmission T_{ab} [52],

$$P(T_{ab}) = \int_0^\infty \frac{dT_a}{T_a/N} \exp\left(-\frac{T_{ab}}{T_a/N}\right) P(T_a) \,. \qquad (26)$$

This equation relates the moments of total transmission and one channel transmission according to $\langle T_{ab}^n \rangle = n! \langle T_a^n \rangle$. One concludes immediately that any finite ΔT_a necessarily implies a deviation from the law (23). These deviations - of order $1/\langle g \rangle$ - have been observed [51]. When $\langle g \rangle = 1$, fluctuations in T_a become as large the average. In Chapter 4 we shall discuss that $\langle g \rangle = 1$ marks the onset of localization in a finite, open system. Far in the extended regime $\langle g \rangle \gg 1$, we expect both T_a and g to be normally distributed. Yet, a rigorous general theory that is also able to include the above long range correlations, is still absent. Only a perturbation theory for T_a exists [53], and a nonperturbational theory for broken time-reversal [54].

The distribution of transmission eigenvalues $\rho(T)$ in the extended regime was given by Mello and Pichard [55], using random matrix theory. In the limit $N \to \infty$,

$$\rho(T) = \frac{\ell^*}{L} \frac{1}{2T\sqrt{1-T}} \qquad \text{for } 4e^{-2L/\ell^*} < T < 1 \,. \qquad (27)$$

This distribution implies that $\langle T \rangle = \ell^*/L$, i.e. the channel average equals the ensemble-average given by Ohm's law (18). It also implies that many channels have *exponentially* small transmission, i.e. are "closed", whereas the rest has $T \approx 1$, i.e. are open.

2.3.4. *Open Chaotic Cavities*
Today, considerable attention is devoted to the analogies between disordered systems and chaotic cavities, stimulated by the many successes achieved for both. Though physically entirely different, conceptually they look very similar. In both cases, the transmission eigenvalue distribution $\rho(T)$ as well as transmission fluctuations $P(T_{ab})$ play a crucial role. For instance, for a chaotic system with N reflection channels and N transmission channels, the equivalent of Eq. (27) is [56]

$$\rho(T) = \frac{1}{\pi\sqrt{T}\sqrt{1-T}} \,, \qquad (28)$$

which has less weight at small transmission values as compared to disordered wires in the extended regime. As a result, the mean $\langle T \rangle = 1/2$ is much bigger.

A second difference concerns the eigenvalue repulsion. For chaotic systems the Dyson ensembles predict that

$$P(T_1, T_2) \sim |T_1 - T_2|^{\beta} = \exp\left(-\beta \log |T_1 - T_2|\right). \tag{29}$$

i.e. the probability that two different channels have the same transmission is small. This implies the existence of strong channel correlations, just as we have seen in the extended regime of disordered systems. The eigenvalue repulsion is often written as an exponential in order to make contact with Gibb ensembles in statistical mechanics. For chaotic systems the pair potential $-\log |T_1 - T_2|$ responsible for repulsion is seen to be logarithmic.

For disordered systems, the pair potential has been calculated by Beenakker and Rejaei [57] for the case $\beta = 2$ (broken time reversal). They concluded that it coincides with the one for chaotic cavities only when $T_1, T_2 \approx 1$. For small transmission however, the "pair potential" approaches $-\frac{1}{2}\log |T_1 - T_2|$, i.e. a factor 2 smaller but still logarithmic. This result looks innocent and academical. One important consequence is that the universal conductance value in a chaotic cavity is not equal to the one ($\Delta g = \sqrt{2/15\beta}$) calculated for disordered wires in the extended regime. For an open chaotic billiard one finds the somewhat smaller value [21]

$$\Delta g = \sqrt{1/8\beta}. \tag{30}$$

2.3.5. *Statistics of Transmission in Localized Regime*

Pichard and Sanquer [83], followed by Van Langen, Brouwer and Beenakker [54] predict both channel transmission and total transmission to be distributed log-normally in the localized regime, i.e.

$$P(T) = \frac{1}{\sqrt{4\pi T}}\frac{\xi}{L}\exp\left[-\frac{\xi}{4L}\left(\frac{L}{\xi} + \log T\right)^2\right]. \tag{31}$$

Here ξ is the localization length and in random matrix theory given by $\xi = \beta N \ell^*/2$, i.e. essentially the number of modes times the transport mean free path. I recall that the localized regime is defined by Eq. (20), which leads to $N\ell^* > L$. From Eq. (31) it follows that

$$\langle \log T \rangle^2 = -2 \langle \log T \rangle. \tag{32}$$

As a result $-(\log T)/L$ converges "almost surely" to $1/\xi$ as the length of the sample exceeds the localization length. This notion provides a crucial difference between absorption and localization, who both yield an exponentially small average transmission. In the localized regime the lognormal distribution predicts a peak at a very small value but, in sharp contrast to absorption, a very significant tail towards large values. It is interesting to note that by breaking time-reversal symmetry ($\beta = 1 \rightarrow 2$), the localization length actually doubles. The distribution of the conductance g has also been argued to become lognormal [58, 21], implying that the universal conductance fluctuations (25) disappear in the localized regime.

The transmission eigenvalue distribution $\rho(T)$ in the localized regime is predicted to be quite different from the one in the extended regime. It is convenient to introduce the variable x according to $T = 1/\cosh^2 x$ so that, according to Eq (27), x is uniformly distributed between 0 and $L/\ell^* \gg 1$ in the extended regime. In the localized regime the distribution $\rho(x)$ is predicted

to "crystallize": It exhibits maxima at periodic positions of x [21]. In this respect the transition from the extended to the localized regime looks like a liquid-solid phase transition.

2.3.6. Ab initio studies

Ab-initio studies try to solve a "real" disordered system numerically exactly. Consider the scattering of an incident plane wave $\psi_0(\mathbf{r})$ from M identical scatterers, with scattering matrix t, distributed randomly in d dimensions. In the simplest case that these can be represented by point particles (a severe simplification), the solution of this problem requires the diagonalization of a complex-valued $M \times M$ matrix. In that case, the equation to be solved is [59]

$$\psi(\mathbf{r}_i) = \psi_0(\mathbf{r}_i) + t \sum_{j \neq i}^{M} G(\mathbf{r}_{ij}) \, \psi(\mathbf{r}_j) \,. \tag{33}$$

The matrix $G(\mathbf{r}_{ij})$ describes a scattered spherical wave emitted by scatterer i and propagating to scatterer j. One wishes to solve this equation in a medium with typical size L and typical density M/L^d. To guarantee multiple scattering, the system size must exceed the mean free path ℓ. Assuming that $\ell \approx 1/n\sigma$ and assuming optimal scattering $\sigma \sim \lambda^{d-1}$ one learns that $L < \lambda \, M^{1/(d-1)}$. In 3D and for $M = 1000$ one can consider a medium which is at most 30 times bigger than the wavelength. Of resonance, in particular for pure Rayleigh scattering, the maximal size is even smaller.

Equation (33) can be written as [60, 61]

$$\boldsymbol{\Psi}(\{\mathbf{r}_i\}) = \mathbf{M}(\{\mathbf{r}_i\})^{-1} \cdot \boldsymbol{\Psi}_0(\{\mathbf{r}_i\}) \,, \tag{34}$$

where $\{\mathbf{r}_i\}$ stands for one realization of the scatterers, $\mathbf{M} = \mathbf{I} - t\mathbf{G}$ and $\boldsymbol{\Psi}$ is the M-dimensional vector containing the wave function at the scatterer positions. From this equation it is clear that an eigenvalue of $\mathbf{M}$ close to zero corresponds to a large scattered field, typically the case for a resonant frequency. An eigenvalue exactly equal to zero would actually correspond to a localized state, that would even persist in the absence of incident field. However, scattering outside the system prevents states to be really localized. The clustering of eigenvalues near zero in a certain frequency interval may thus be a signature of localized states.

The t-matrix of the particles depends on frequency. By energy conservation (optical theorem) it has the form $t = c[\exp(2i\phi) - 1]$ with c a real-valued constant and $\phi(\omega)$ the scattering phase shift. The eigenvalues of $\mathbf{M}$ and $\mathbf{G}$ are related by $\lambda_{\mathbf{M}} = 1 - t\lambda_{\mathbf{G}}$. An eigenvalue $\lambda_{\mathbf{G}}$ with $\mathrm{Re}\,\lambda_{\mathbf{G}} = -1$ allows exactly one choice for ϕ such that $\lambda_{\mathbf{M}} = 0$, the choice depending on $\mathrm{Im}\,\lambda_{\mathbf{G}}$. The frequency associated with this phase shift would then correspond to a localized state.

In Figure 3 we show the outcome of numerical work done by Rusek, Orlowski and Mostowski [60, 61]. It shows contour plots of the density of eigenvalues $\lambda_{\mathbf{G}}$ for randomly positioned point cylinders i.e the numerical solution of Eq. (33) in 2D. It can be seen that as the size of the system grows, eigenvalues start to cluster more and more around the point $\mathrm{Re}\,\lambda_{\mathbf{G}} = -1$, the convergence being fastest for $\mathrm{Im}\,\lambda_{\mathbf{G}} = 0$. This value corresponds to a phase shift $\phi = \pi/2$ i.e. resonating scatterers. In principle one would expect all states to become localized in 2D, even the ones with very small scattering phase shift ϕ (which in Fig. 3 would correspond to $\mathrm{Im}\,\lambda_{\mathbf{G}} \gg 1$). But to see that one has to increase the system - at constant number density - to astronomical lengths.
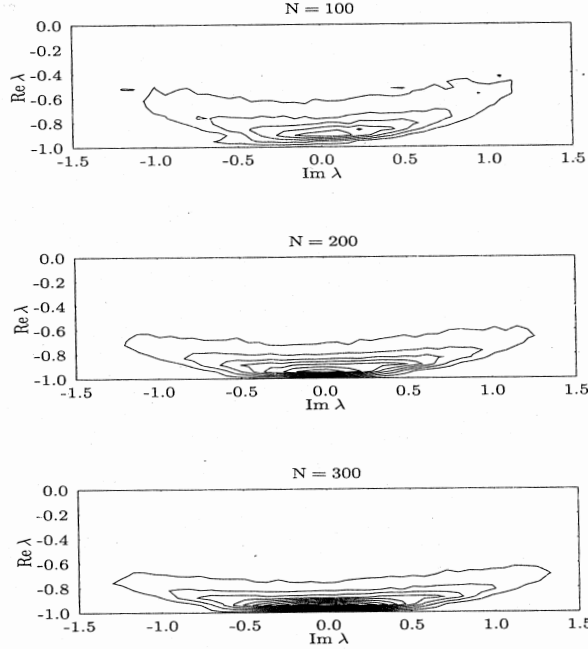
Figure 3. Contour plot of the density of complex-valued eigenvalues λ_G calculated for 10^3 different distributions of randomly positioned point cylinders. The number of particles N increases from top to bottom, at constant number density n, chosen to be one scatterer per wavelength squared. The clustering of eigenvalues near $\text{Re}\,\lambda = -1$ marks the onset of localization for the infinite system. Taken from Rusek, Orlowski and Mostowski [61], with kind permission from the authors.

3. Anderson Model

The simplest model that exhibits basic features of localization is the Anderson model. This model describes the hopping of a tightly bound electron from one site to the other. Disorder is introduced by a random potential ε_n at each site, so that the Hamiltonian becomes,

$$H = \sum_{n \in S} \varepsilon_n \, |n\rangle \langle n| + V \sum_{n,m \in S}' |n\rangle \langle m| \ . \tag{35}$$

$E = 0$ corresponds to the tightly bound state in the absence of quantum tunneling $(V = 0)$. The set S indicates the location of the sites $|n\rangle$ in real space, which can e.g. be Z^d for the simple cubic lattice in d dimensions. Usually, the summation $\sum_{n,m}'$ runs over nearest neighbours only. This kinetic "off-diagonal" part describes the hopping from one site to the other, and is in the standard Anderson model equal to a deterministic hopping matrix element V. Also models with off-diagonal disorder have been discussed, where V is random rather than ε, modeling situations with either broken time-reversal symmetry or spin-orbit scattering. They are believed to be part of a different universality class, characterized by different critical exponents at the transition [62, 63], and sometimes even different transitions.

A crucial property in the model is the number of nearest neighbours Z. The simple cubic lattice in dimension d $(Z = 2d)$ is the simplest choice, but - despite its exotic structure - also the Cayley tree (or Bethe Lattice, see Figure 4) with connectivity K $(Z = K + 1)$ is of interest
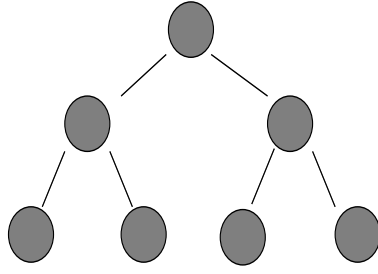
18

Figure 4. The Bethe lattice or Cayley tree, here drawn with connectivity $K = 2$.

because for this geometry, like in one dimension, the Anderson model can be solved exactly. In both cases, the spectrum of the regular system ($\varepsilon = 0$) is determined by the number of nearest neighbours Z and the amount of quantum tunneling V: $E \in [-ZV, ZV]$. The site potentials ε_n are chosen to be statistically independent, identically distributed random variables, and it is common to choose $P(\varepsilon) = 1/W$ for $-W/2 < \varepsilon < W/2$ ("the rectangular diagonal Anderson model"), although also a Cauchy distribution or a Gaussian distribution for ε is frequenctly encountered.

The Anderson model with diagonal disorder and its numerical solutions have been reviewed recently in the book by Ping Sheng [64]. Fröhlich and Spencer [20] proved strong localization for the Anderson model with diagonal disorder on a cubic lattice in any dimension, in the sense of criterion (4), provided that either the disorder W is sufficiently large, or, for any disorder, at energies sufficiently outside the spectrum of the regular model: $|E - 2dV| \gg 0$. This proof was later refined by Aizenman and Molchanov [65].

3.1. ANDERSON MODEL IN 1D

The extensive literature justifies to spend a few extra words on the one dimensional Anderson model. Some people consider 1D localization not interesting because there is no mobility edge. The critical behavior at the mobility edge is often considered as the most important aspect of Anderson localization.

The study of 1D localization goes back to the early sixties where random matrix techniques by Furstenberg and Osseledec were first applied. The basic starting point is the product of n statistically independent random transfer matrices, $\mathbf{M}_1(E) \cdot \mathbf{M}_2(E) \cdots \mathbf{M}_n(E)$. Transfer matrices typically show up in one dimensional problems and are characterized by $\det \mathbf{M}_n = 1$. The eigenvalue equation for the 1D Anderson model can be written as,

$$V (\psi_{n-1} + \psi_{n+1}) + \varepsilon_n \psi_n = E \psi_n, \tag{36}$$

where $\psi_n = \langle n | \psi_E \rangle$ is the eigenfunction at site n. Note that $\psi_{n+1} + \psi_{n-1} - 2\psi_n$ is a discrete version for the Laplacian in 1D. Equation (36) is equivalent to,

$$\begin{pmatrix} \psi_n \\ \psi_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & (E - \varepsilon_n)/V \end{pmatrix} \cdot \begin{pmatrix} \psi_{n-1} \\ \psi_n \end{pmatrix}, \tag{37}$$

where the matrix is identified as $\mathbf{M}_n$. Evidently, ψ_n for large n is determined by a product of the kind $\mathbf{M}_1(E) \cdot \mathbf{M}_2(E) \cdots \mathbf{M}_n(E)$.

Furstenberg's theorem says that the norm of this matrix is, for "almost every" realization, proportional to $\exp[n\gamma(E)]$, where for any energy E, the Lyapunov exponent $\gamma(E) > 0$. A simple proof, though restricted to 2×2 matrices has been given by Delyon, Kunz and Souillard [66]. Ossedelecs theorem says that if $\gamma(E) > 0$, for" almost every" realization of the system, one *unique* vector $\psi_0(E)$ exists such that the product $\mathbf{M}_1(E) \cdot \mathbf{M}_2(E) \cdots \mathbf{M}_n(E) \cdot_0$ decays as $\exp[-n\gamma(E)]$. Note that, by Furstenberg's theorem applied to $\mathbf{M}_n(E)^{-1} \cdots \mathbf{M}_2(E)^{-1} \cdot \mathbf{M}_1(E)^{-1}$, there is at least one such a vector.

Osseledec's Theorem excludes - for almost all realizations - a solution of the 1D Anderson model to be part of the absolute continuous spectrum, since solutions either decay or grow exponentially. The solution at energy E will be localized when the vectors $\psi_0^{\pm}$ at some site for left- $(n \to -\infty)$ and righthand $(n \to \infty)$ decay coincide, assuring exponential decay at both sides. If not, E is not part of the spectrum. The subtle point here is that the set of realizations with probability zero for which Osseledec's Theorem does not hold, depends on the energy. Statements for almost all realizations and all energies cannot be made. As a result one can only state that "for almost all realizations", and then for "almost any" energy, the 1D Anderson model has a pure point spectrum and is thus localized. It is still possible that the whole spectrum coincides with the set of measure zero that we don't know anything about! One example is by simply putting $\varepsilon_n = \lambda \cos(2\pi\nu)$, with ν some irrational number and $\lambda > 2$. Except for exotic potentials, the 1D Anderson model is - under very broad conditions - localized in the sense of definition (4).

We shall briefly state some major findings of the 1D Anderson model. One of the most important fundamental results - formulated first by Thouless [67] - is the Kramers-Kronig relation between Lyapunov exponent $\gamma(E)$ and density of states (per site) $\rho(E)$, showing that $\gamma(E) + i\rho(E)$ is actually an analytic function when continued in the complex sheet $\mathrm{Im}\, E > 0$. Kappus and Wegner [68] and Lambert [71] showed that the Lyapunov exponent near the band center $E = 0$ takes the form,

$$\gamma(E) = \frac{\langle \varepsilon^2 \rangle}{V^2} F\left(\frac{EV}{\langle \varepsilon^2 \rangle}\right) \quad (W, |E| \ll V). \tag{38}$$

with $F(x)$ a smooth function, obeying $F(0) \approx 8.7537$ and $F(x \to \pm\infty) = 8$. The result $\gamma = 8\langle \varepsilon^2 \rangle / V^2$ coincides with the perturbational result first obtained by Thouless [69]. The presence of the function $F(x)$ denotes that perturbation theory breaks down for $|E| < \langle \varepsilon^2 \rangle / V$, which is called an *anomaly*. The anomaly is weak and the localization length basically scales as $\xi(E) \approx V^2 / 8 \langle \varepsilon^2 \rangle$.

Near the band edges $E = \pm 2V$ of the regular system one obtains a rather nontrivial scaling relation as $\langle \varepsilon^2 \rangle \ll V^2$,

$$\gamma(E) = \frac{\langle \varepsilon^2 \rangle^{1/3}}{V^{2/3}} H\left(\frac{V^{1/3}(|E| - 2V)}{\langle \varepsilon^2 \rangle^{2/3}}\right). \tag{39}$$

The function $H(x)$ has been explicitly calculated by Derrida and Gardner [70], and satisfies $H(0) = \sqrt{\pi} 6^{1/3} / 2\Gamma(\frac{1}{6}) \approx 0.2893$, $H(x \to \infty) = \sqrt{x}$ and $H(x \to -\infty) = -1/8x$. At the band edge we thus find the nonanalytic scaling $\gamma(E) \sim (\delta\varepsilon/V)^{2/3}$. The scaling relations (38) and (39) can actually be derived using the elegant renormalization procedure developed by Bouchaud and Daoud [72].

It is worth mentioning that for the Cauchy distribution, the Anderson model with diagonal disorder allows for an exact solution of the Lyapunov exponent for all energies and for all
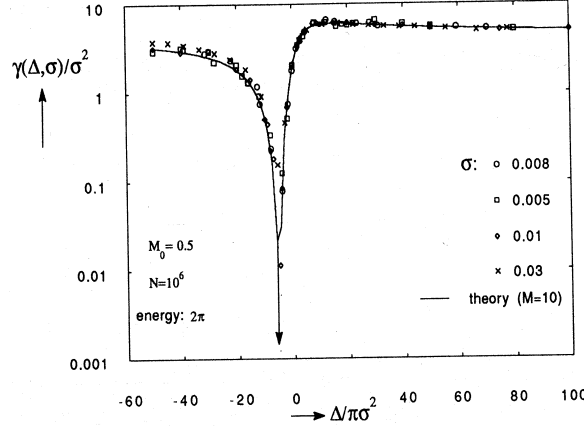
Figure 5. Numerical simulation of the Lyapunov exponent in a stack composed of 10^6 random dielectric layers with a contrast $m_0 = 0.5$ in their index of refraction. Δ denotes the distance to the band center of a transmission band; σ is the standard deviation of the fluctuations in the dielectric constant. The solid line denotes the prediction of degenerate perturbation theory. Taken from Van Tiggelen and Tip [74].

disorders. This model is often called the Lloyd model. Assuming the distribution to be $P(\varepsilon) = W^2/\pi(W^2 + \varepsilon^2)$, one finds [73]

$$\gamma(E,W) = \log|z(E,W)| \,, \ \text{with} \ z + \frac{1}{z} = E + iW \,. \tag{40}$$

which exhibits a different scaling $\gamma = \sqrt{W/2}$ on the band edges, as compared to distributions with finite variance.

The 1D Anderson model has an analogy in the scattering of classical waves from random layers, where the relations (38) and (39) have been seen to apply. The anomalies in the band center of the optical model are much more pronounced, leading sometimes to giant localization lengths [74]. In Fig. 5 I show the anomaly function $F(x)$ defined by Eq. (38), calculated light propagation in a random dielectric stack. The band center is here chosen to be right in the middle of a pass band. The scaling (38) is confirmed for small fluctuations in the dielectric constant, with an almost vanishing Lyapunov exponent in the band center. The solid line denotes the prediction of a modified version of the perturbation theory by Lambert [71]. The agreement is very good.

It has been shown that 1D localization of waves is unstable against small isotropic scattering [75]. This is bad news for seismology, where 1D localization has been investigated in view of the layered structure of the Earth crust [10]. Localization of light in 1D layered media has also been discussed in relation with gain [76]. It turned out that gain, just like absorption, *enhances* the Lyapunov exponent, somewhat surprising since gain may have been expected to enhance transmission.

3.1.1. *Kicked Rotor*

The Anderson model in 1D has seen an interesting application in the quantum mechanical treatment of a kicked rotor. The classical picture of a rotor with moment of inertia I that is kicked abruptly at times t_n with a torque $\partial V/\partial \theta$ is easy. The equations of motion for the classical rotor after kick $\sum_n \Delta t \, \delta(t - t_n)$ easily show that the angular momentum $J_n = J(t_n + 0)$ will finally increase in time as,

$$\lim_{n \to \infty} \frac{J_n^2}{n} = (\Delta t)^2 \left\langle \left(\frac{\partial V}{\partial \theta} \right)^2 \right\rangle . \tag{41}$$

This looks like a diffusion law of the kind (2), the angular momentum J_n taking over the role of displacement $\mathbf{r}$.

The quantum mechanical version of the kicked rotor is given by the following time-dependent Schrödinger equation ($\hbar = 1$),

$$i \frac{\partial \psi(\theta, t)}{\partial t} = \frac{1}{2I} \frac{\partial^2 \psi(\theta, t)}{\partial \theta^2} + V(\theta) \Delta t \sum_n \delta(t - t_n) \psi(\theta, t) . \tag{42}$$

Following Fishman, Grempel and Prange [11] we set $t_n = n t_0$, i.e. the rotor is regularly kicked. As a result of this periodicity, the solution can be written as a superposition of "eigenfunctions" $\psi_\omega = \exp(-i\omega t) u(\theta, t)$, with $u(t) = u(t + t_0)$. We call ω a quasi eigenfrequncy because $u(t)$ still depends on time. Let $U(\theta) = \frac{1}{2}[u(\theta, t_n - 0) + u(\theta, t_n + 0)]$. A simple closed equation can be obtained for the Fourier transform of $U(\theta)$,

$$U_k = \frac{1}{2\pi} \int_0^{2\pi} d\theta \, e^{ik\theta} \, U(\theta) \tag{43}$$

which is nothing more than Fourier component of the eigenfunction corresponding to a quantized angular momentum $J_k = k$, which reads

$$\sum_{l \neq 0} V_l U_{k+l} + \varepsilon_k U_k = E U_k , \tag{44}$$

This equation looks like the 1D Anderson model (36). The "random site potential" is given by $\varepsilon_k = \tan(\frac{1}{2}\omega t_0 - \frac{1}{4}t_0 k^2/I)$. The hopping element V_l is given by the Fourier transform (43) applied to $\tan[\frac{1}{2}\Delta t V(\theta)]$. Finally, the "energy" $E = -V_0$. Compared to the standard Anderson model, Eq. (44) allows for "hopping" (discrete transitions in the angular momentum) beyond nearest neighbours. Only the somewhat special potential $V(\theta) = (2/\Delta t)\arctan(v \cos \theta - w)$ has nearest neighbour hopping only, with strength v.

Without kicking ($V =$ constant), quasi eigenfrequencies $\omega_k = -\arctan w + \frac{1}{2}k^2 t_0/2I$ mod(2π) exist with eigenfunction $u_n = \delta_{nk}$ "localized" at site k, i.e. with constant angular momentum k. In the presence of kicking we notice that, during the kicking from one site to the other, the variable $\frac{1}{2}\omega t_0 - \frac{1}{4}t_0 k^2/I$ (mod π) becomes a pseudo-random variable uniformly distributed between $-\pi/2$ and $\pi/2$ provided that $t_0/4\pi I$ is an *irrational* number. As a result, ε_k is pseudo-random with a Cauchy distribution and the model resembles the Lloyd model discussed above. We may expect all wave functions to localized in n space, with a localization length given by Eq. (40). Localization occurs in angular momentum space, i.e. the probability of finding high angular momenta is exponentially small *at all times*. Hence the name "dynamical localization". This statement clearly violates the classical result (41). The validity of the Lloyd model for the exponential decay of the wave function has been verified [11].

The localization principle tells us that the quasi eigenfrequency spectrum is discrete. All localized states that overlap a given site k are separated in frequency. Assuming the rotor to be at rest at $t = 0$, one expects the time dependent wave function,

$$\psi_k(t) = \sum_n \exp(-i\omega_n t) U_k^{(n)} \bar{U}_0^{(n)}, \qquad (45)$$

to be a superposition of some countable set of incommensurate frequencies, making it quasi periodic in time. As a result, the probability of finding the oscillator at the original site, i.e. with the same kinetic energy is finite at any time. In section 2.1 a finite "return probability" was argued to be a hallmark for localization.

A number of questions come up by this analogy. The first is mathematically interesting but difficult to answer in general: What kind of "pseudo-randomness" in the Anderson model is sufficient to make it equivalent to one with real randomness? The second question touches the heart of modern quantum mechanics. By the correspondence principle, the classical solution must be some limit of quantum mechanics. Yet, the last one gives non-diffuse behavior, whereas the first yields diffuse behavior for the angular momentum of the kicked rotor. The solution of this "coherence" problem is believed to reside in the coupling of the quantum object with the environment. As this coupling increases, the classical solution is believed to apply more and more.

Ammann etal. recently reported the observation of decoherence in a gas of ultracold cesium atoms, subject to a pulses wave of light [77]. The model of the quantum kicked rotor applies here. The decoherence is in this case caused by the spontaneous emission, which destroys the phase of the Cesium atoms. A significant modification to Eq. (41) was observed as the amount of spontaneous emission decreased. Dynamical localization in more than one dimension has also been reported in literature [78].

3.2. ANDERSON MODEL IN 2D AND 3D

The original paper of Anderson dealt with the Hamiltonian (35) assuming many nearest neigbours, and a rectangular distribution $P(\varepsilon) = 1/W$ for $-W/2 < \varepsilon < W/2$. Anderson showed that eigenstates become localized for sufficiently large value for W/V. The critical value for W/V increases with the number of nearest neighbours, implying that localization becomes more and more difficult as the dimension of the system increases. The dependence of localization upon dimensionality is thus very basic.

We now know that the Anderson model is fully localized in one and two dimensions ($Z = 2$ and $Z = 4$) for *any* value of W/V. Only the off-diagonal Anderson model with spin-orbit scattering seems to have a transition in 2D [62]. For $d > 3$ the spectrum exhibits a mobility edge E_c separating two regimes with exponentially localized and extended eigenfunctions, separated by a "mobility edge". This mobility edge was first suggested by Mott. The localized states start to appear at the edges $E = \pm ZV$ of the spectrum, and move gradually towards the band center as the disorder increases. In Fig. 6 we show the phase diagram on the basis of a sophisticated numerical solution of the Anderson model on a simple cubic lattice in three dimensions. It can be inferred that the extended regime completely disappears for $W/V > 16.5$ (or equivalently $\Delta\varepsilon/V > 4.7$), which is not far from the value of 26 estimated by the back-of-the-envelope calculation of Anderson. For the Cauchy distribution $P(\varepsilon) = (1/\pi) W/(\varepsilon^2 + W^2)$ the critical value in 3D is $W/V = 3.8$ whereas the Gaussian distribution has $W/V = 20.9$ [79].

An interesting consequence of the random potential is the formation of localized states outside the spectrum of the regular system. This phenomenon has been studied extensively, and
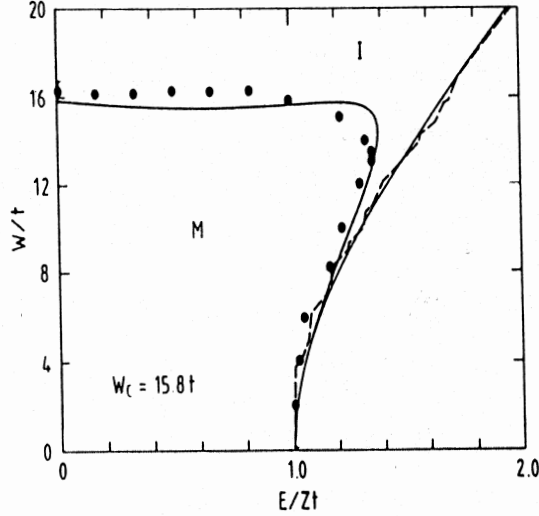
Figure 6. Phase diagram of the rectangular Anderson model with diagonal disorder on a 3D cubic lattice. The solid triangles are exact numerical results for the mobility edge. The solid line denotes the outcome of the selfconsistent theory of localization, discussed in Chapter 5. The thin solid line denotes the prediction of effective medium theory for the shift of the band edge beyond $E = 6V$. Taken from Vollhardt and Wölfle [7], with kind permission from the authors.

it has turned out that the density of states $\rho(E)$ follows a universal curve $\rho(E) \sim \exp[-|E - E_b|^{-d/2}]$, called a Lifshitz tail [24, 25], provided the spectrum is bounded (with bound E_b i.e. the random potentials ε are bounded. This is not true if their distribution is Gaussian. In that case one finds [24, 79] $\rho(E) \sim \exp[-C|E|^{2-d/2}]$. In 1D, one finds more precisely $\rho(E) \sim |E|^{3/4}W^{-5/3})\exp[-C|E|^{3/2}/W^2]$ [70].

In Figure 7 we show the fractal dimension β, defined in Eq. (11), for the 2D Anderson model on a simple 2D square lattice. In the center of the band ($E = 0$) β is somewhat smaller than 2, indicating that these states are nearly extended. Beyond the band edge $E = 4V$ of the regular system, the exponentially localized Lifshitz tail makes that $\beta = 0$, even at modest disorders. As the disorder W increases, the localization length decreases, and the appearance of exponentially localized states ($\beta = 0$) becomes more and more evident. In 2 dimensions, in principle all states are exponentially localized, but the localization length ξ may be macroscopically large. Economou etal. [82] give the approximate formula,

$$\xi(E = 0) = 2.72\ell(0)\exp\left(\frac{\sqrt{2}\pi\ell(0)}{a}\right) \tag{46}$$

for the localization length in the band center in terms of the mean free path ℓ and the lattice spacing a. In 3D one finds a fractal dimension $\beta = 1.54 \pm 0.08$ at the mobility edge [80].

As mentioned already in section 2.1, the level spacing distribution function is sensitive to localization. For the 3D Anderson model it has explicitly been verified that the distributions (13) and (14) apply in the extended regime, respectively with and without broken time reversal, whereas a totally uncorrelated (Poisson) distribution emerges in the localized regime. At the critical disorder $W = 16.5\,V$ a critical distribution shows up [84],
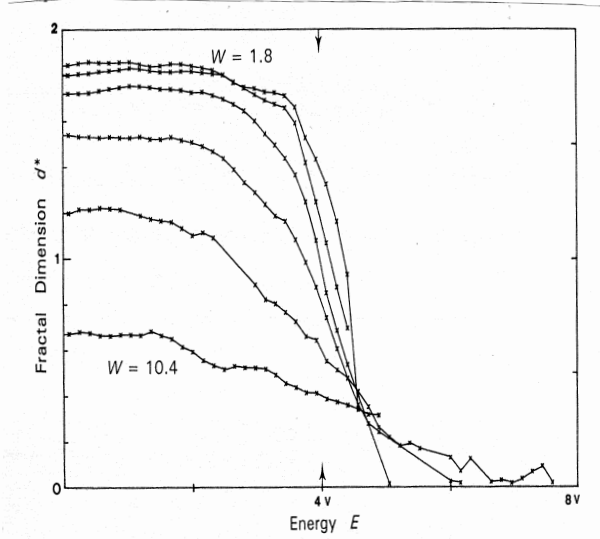
Figure 7. Fractal dimension, defined in Eq. (11) of the 2D Anderson model with diagonal disorder, as a function of energy. $E = 4V$ is the band edge of the model without disorder. Taken from Schreiber [27], with kind permission from the author.

$$P(s) \approx As \exp(-Bs^{1.25}) \,, \tag{47}$$

that seems to be independent on the presence or absence of time reversal symmetry. Near the mobility edge the critical exponent of the localization length is inferred to be $\nu \approx 1.4$ [81], largely independent of the chosen distribution for ε_n and significantly different from the mean field value $\nu = 1$ obtained by approximate (mean field) theories. Surprisingly, this critical exponent is seen to be left unchanged if time-reversal is broken [84, 85].

3.3. CAYLEY TREE

The Anderson model on the Cayley tree has first been discussed by Abou-Chacra, Anderson and Thouless [86], with important contributions later by Kunz and Souillard [87] and Kawarabayashi and Suzuki [88]. It was realized by Abou-Chacra etal. that a significant approximation made in the original Anderson paper necessary to solve the tight-binding problem, is actually not an approximation for the Cayley tree. This means that the original argument by Anderson for localization is rigorous for this system. The Cayley tree is special because it has no closed paths and the sites on the boundary are as numerous as inside the system. In comparing the number of nearest neighbours $K + 1$ to the number $2d$ for a simple cubic lattice, the connectivity K can be identified as an effective dimension d by means of the relation $K + 1 = 2d$.

It can be shown that the exponential decay rate $\gamma(E)$ of the Green's function (in units of the lattice spacing), defined as,

$$\gamma(E) = -\lim_{j \to \infty} \frac{\log|G(0, j, E)|}{j} \,, \tag{48}$$

is self-averaging and given by,

$$\gamma(E) = -\log|V| + \left\langle \log|G_>(i, i, E)|^{-1} \right\rangle . \tag{49}$$

The symbol $>$ indicates that the return Green's function $G(i, i)$ is calculated without counting paths on earlier levels of the tree. The absence of closed paths on the Cayley tree guarantees that the $G_>(i, i)$ with i on the same level of the tree are all mutually independent. This notion enables to solve for the whole distribution of $\tilde{G}(i, i)$, and to evaluate the decay (49) of the Green's function straightforwardly.

However, on the Cayley, Eq. (48) does not yet guarantee strict localization. The number of sites at distance L grows exponentially with L according to $(K + 1)K^{L-1}$, and not algebraically L^{d-1} as in Euclidean space. In order to have strict localization, the exponential decay of the Green's function must overcome this exponential growth. Hence, the criterion for localization becomes [88],

$$\frac{1}{\xi(E)} \equiv \gamma(E) - \log K > 0 . \tag{50}$$

This immediately confirms the conjecture of Anderson that localization becomes more difficult as the dimensionality $K \sim d$ increases. It can be shown that the critical exponent of the localization length $\xi(E)$ near the mobility edge, is unity, i.e. $\xi(E) \sim |E - E_c|^{-1}$.

The calculation of $\gamma(E)$ looks very much like the one of the Lyapunov exponent of the one dimensional Anderson model discussed earlier in Eq. (40). In fact, for $K = 1$ it *is* exactly the 1D Anderson model. Just like in 1D, an analytical solution for $\xi(E)$ exists, valid for all energies and any disorder , for a Cauchy distribution of the potential ε, $P(\varepsilon) = W/\pi(\varepsilon^2 + W^2)$. By criterion (50) the mobility edge separating localized and delocalized states is given by [88]

$$\frac{E^2}{(K + 1)^2} + \frac{W^2}{(K - 1)^2} = V^2 . \tag{51}$$

In particular, all states become localized when $W/V > K - 1$, a conclusion that can be compared qualitatively to Anderson's "best estimate" (Figure 3 of Ref. [14]). For $K = 1$ all states are localized for any disorder, as we know already from other work. For $K = 3$, the equivalent of two dimensions on the Cayley tree, we notice the existence of a mobility edge, contrary to the 2D Anderson model on a cubic lattice.

In Fig. 8 we show a comparison of the phase diagram for the Anderson Model on the Cayley tree ($K = 5$) and the one on a 3D simple cubic lattice, obtained by finite size scaling [79], both using a Cauchy distribution of the site potentials. The agreement is remarkable. In higher dimensions the agreement is expected to be even better.

4. Great Principles

Simplified models give a good impression of what is going on and when localization can be expected to occur. More realistic models, hopefully closer to experiments, are hardly ever exactly solvable. The question is whether we can formulate "great principles" that provide basic features of localization, and that apply to more complicated systems as well, such as ones with absorption, with short or long-range correlations, anisotropic and continuous models (described by spatial and/or time-dependent correlation function rather than with discrete random potentials). This section addresses three great principles.
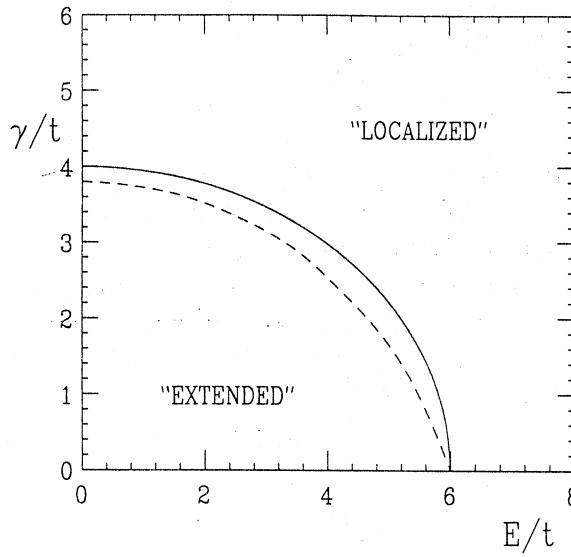
Figure 8. Mobility edge trajectory obtained for a Anderson model with diagonal Cauchy disorder. Solid line line is Eq. (51) for the Cayley tree with $K = 5$, dashed is the exact numerical solution for the 2D cubic lattice. Taken from Kawarabayashi and Suzuki [88], with kind permission from the authors.

4.1. IOFFE-REGEL CRITERION

The mean free path (or extinction length) ℓ denotes a typical length scale of a particle or wave packet between two subsequent collisions. For a low number density n of particles with total scattering cross-section σ one derives the familiar expression $\ell = 1/n\sigma$. A long mean free path denotes that "disorder is weak".

The important length scale for interference effect is - of course - the wavelength $\lambda = 2\pi/k$. What happens if the mean free path becomes comparable to the wavelength? It means that interference continues to dominate during scattering. It signals a regime in which scattering can no longer be described classically, and some kind of catastrophe has to occur. This kind of reasoning led Ioffe and Regel to a criterion for localization in disordered semi-conductors,

$$k\ell < \text{constant} \approx 1\,, \tag{52}$$

now called the Ioffe-Regel criterion. Actually, this reasoning is only a very small part of the extensive paper by Ioffe and Regel [89]. In my opinion, their intention was more to identify the "strongly scattering" regime, where one cannot longer identify multiple scattering with a subsequent propagation of plane waves, but rather one with overdamped waves. The criterion for this regime look like a Ioffe-Regel criterion: $k\ell < 0.5$ [90]. Equation (52) seems to state that localization and "strong scattering" are both complicated, and must therefore be related. It was who Mott who really made the link between the Ioffe-Regel criterion (52) and localization of waves. He also estimated the constant in Eq. (52) to be of order unity.

Many physicists have criticized the Ioffe-Regel criterion. Their main objection is the fact that ℓ is a length scale of the field, according to Eq. (10), whereas localization is (at least) a property of the intensity, as indicated in Eq. (9). In spite of these arguments, the Ioffe-Regel criterion has been verified to hold quite accurately in the 3D Anderson model, the constant ranging from 0.8 to 1.0 [64]. In addition, it also emerges from approximate theories such as the self consistent theory of localization [15, 16] and quantum field theories [19]. It is now generally accepted as an approximate criterion for localization of waves in infinite 3D random media. I stress that it is not a *universal* criterion, in the sense that the constant is known to vary from one model to the other, and perhaps even from experiment to experiment, as opposed to critical exponents around the transition [91].

It is important to note is that the mean free path figuring in the Ioffe-Regel criterion is the *scattering* mean free path (extinction length), and not the transport mean free path ℓ^*. I recall that the latter is defined by means of the relation $D = v_E \ell^* / d$. The DC electronic conductivity can easily be shown to be

$$\sigma \approx \frac{1}{d}(e^2/h)\ell^* k^{d-1} \, . \tag{53}$$

If we accept definition (1) as then $\ell^* = 0$ at the mobility edge. The velocity v_E is not believed to vanish.

The confusion of transport and scattering mean free path can lead to erroneous conclusions, of which I mention one of historical importance. In his discussion of the Ioffe-Regel criterion, Mott concluded that if $\sigma = 0$ in the localized regime, and finite in the extended regime given by criterion (52), a *minimum conductivity* $\sigma_{\min} \approx (e^2/h)k^{d-2}$ must exist, suggesting the Anderson transition to be first-order. In 2D this minimum conductivity would even be a fundamental constant. Although this argument still does not rule out a minimum conductivity, it is now widely accepted that the transition is continuous, and that the Mott minimum conductivity does not exist.

4.1.1. *Generalizations*

Several authors have tried to find a microscopic base of the Ioffe-Regel criterion. Both the selfconsistent theory of localization and non-linear field methods suggest that if one accepts Eq. (1) as a working definition for localization, the Ioffe-Regel criterion follows, the constant being somewhat dependent on the model but always of the order of unity. It would be tempting to apply the criterion in dimensions less then three. Yet, we know that in principle all states are localized. Non-linear sigma models are theories that expand around the critical dimension $d = 2$, and extrapolate towards higher dimensions [18, 19]. The outcome is a generalized criterion for d dimensions,

$$k\ell = \left(\frac{1}{2\pi(d-2)}\right)^{1/(d-1)} . \tag{54}$$

This formula confirms that the criterion for localization becomes more severe in higher dimensions, a conclusion that followed already from the work of Anderson, and clearly identifies $d = 2$ as the critical dimension for the presence of a mobility edge.

For the 3D Anderson model, the Ioffe-Regel criterion was derived by Zdetsis, Soukoulis and Economou [82, 92], starting from the self consistent theory of localization, to be discussed later. They noticed that the formula for the mobility edge has a lot in common with the well-known condition for a 3D potential barrier to have a bound state. On the basis of this "potential well analogy" they concluded to mobility edge to be given by,

$$S(E)\ell^2 \approx 8.96 \,, \tag{55}$$

where $S(E)$ is the constant-energy (or -frequency) surface in wave number space. One can assign an average effective wave number $k(E)$ using the relation in free space, $S(E) = 4\pi k(E)^2$, leading to a Ioffe-Regel criterion with a constant equal to 0.844. A completely different model, using randomly distributed point scatterers in 3D yields the constant 0.972 [93].

4.1.2. *Applications: electrons versus classical waves*

What does the Ioffe-Regel criterion learn us? One should first realize that the mean free path ℓ is a complicated function of both energy (or frequency) and disorder. But the mean free path may be expected to decrease with increasing disorder (or increasing impurity density). As a result, the mobility edge occurs at a minimum amount of disorder that depends on k, and thus on energy. This agrees qualitatively with what we know for the Anderson model.

Let us consider an electron at energy $E = k^2/2m$ that propagates in a medium with randomly positioned particles with cross-section $\sigma(E)$ and number density n. For sufficiently low density expects the mean free path to be given by $\ell(E) = 1/n\sigma(E)$. It is well known that the cross-section for ordinary potential scattering is finite at low energies. Hence $\ell(E = 0) < \infty$ for any density n. Since $k \to 0$ at low energies one arrives at the conclusion that all states are localized at sufficiently small energy, i.e near the continuum edge of the regular medium.

Unfortunately, this argument breaks down for classical waves. For light or sound the cross-section at low frequencies has the familiar Rayleigh form, $\sigma(\omega) \sim \omega^4$. As a result $k\ell \sim \omega^{-3} \gg 1$ at low frequencies, and no localization is expected there.

The relation $\ell = 1/n\sigma$ suggests that localization might be possible if the cross-section is large, i.e. at resonances. However, the cross-section can never exceed the "unitary limit" $\sigma \approx \lambda^2/\pi$. Localization may thus be possible for sufficiently high density $n\lambda^3 \approx 1$. The possibility of localization near resonances was first argued in a paper by Sornette and Souillard [94], which was later followed by extensive calculations [95, 96, 97]. One obvious deficiency of the argument is that it is based on a formula for the mean free path that holds only if the number density is small. Both spatial correlations and "dependent scattering" [93] - i.e. recurrent scattering from the same particle - can make the mean free path bigger than anticipated from a low-density extrapolation.

In the band center of the Anderson model $E = 0$ the wavenumber is inversely proportional to the lattice spacing, meaning that localization sets in when the lattice constant exceeds the mean free path, a reasonable conclusion. Near the band edges one finds $k^2 \sim 6V - |E| \to 0$, whereas the mean free path stays finite. This means that localization can set in even when $\ell \gg a$. I note that by Eq. (50) the Cayley tree has its mobility edge, in any dimension larger than one, always determined by the criterion $\ell \equiv 1/\gamma \approx a$.

According to the above, localization seems easier near band edges. It was first suggested by John [5] that periodic structures, subject to small disorder, are very good candidates to observe localization, in particular for classical waves. In a slightly disordered photonic band gap material, localization of light can be expected near the band gap where the density of states of the regular system vanishes, just like electron localization was predicted to occur at $k \approx 0$. The Ioffe-Regel criterion near the band edge ω_b takes the form $(\omega - \omega_b)\ell(\omega)/c_0 \approx 1$. The possibility of light localization still is one of the drivers to manufacture of photonic band gap materials. Recently, exact proofs have been published for classical wave localization - in the sense of criterion (4) - near a frequency gap of the host system [98, 99].

4.2. THOULESS CRITERION

The opening of a disordered system has one important consequence, as first discussed by the classical paper of Thouless [1]. Consider a closed sample of size L, much bigger than the mean free path and in the extended regime, and open it up at two sides in an ideal way, i.e. the boundaries do not prevent in any way the waves to leak through. In the extended regime, the typical time for a wave to traverse the sample is L^2/D. As a result, the eigenstates of the closed system become resonances with typical width $\delta E \approx D/L^2$, called the *Thouless energy*. For De Broglie waves we have assumed that $\hbar = 1$. For classical waves one should look at δE as a typical frequency. Crucial is how the Thouless energy compares to the average level spacing ΔE, which is approximately given by the inverse density of states $1/\rho(E)L^d$. We can define,

$$g \equiv \frac{\delta E}{\Delta E} \approx \rho(E)D(E)L^{d-2}. \tag{56}$$

The dimensionless variable g is called the dimensionless conductance, because the second equality actually makes it coincide with the Drude conductance of the medium.

According to Thouless $g > 1$ implies that the diffusion process can be supported by the microstates of the disordered system implying it to be in the metallic (extended) regime. For $g < 1$, the diffusion process cannot be supported microscopically, because the typical level width necessary for diffusion would contain only one eigenstate, which is insufficient to support a dynamical process. Hence the system is in the localized regime. The criterion for localization in open systems becomes,

$$g = \text{constant} \approx 1. \tag{57}$$

It is instructive to compare the Thouless criterion to the one formulated earlier in Eq. (20) for open systems. For a tube with N transverse modes, length L and mean free path ℓ^*, we argued that localization sets in when $L > \xi$, with the localization length ξ defined by $\xi \approx N\ell^*$. In the extended regime the ensemble -average of the conductance, defined in Eq. (22) is $\langle g \rangle \approx N\ell^*/L$. Hence, near cross-over to the localization regime one recovers the Thouless criterion $\langle g \rangle \approx 1$.

Some find it convenient to rephrase the Thouless criterion in terms of time scales. The time $t_D \hbar/\delta E \sim L^2/D$ is a typical time for an excitation to move through the entire system, whereas $\hbar/\Delta E$, often called the Heisenberg time t_H, represents the maximal time scale possible in the medium. Localization means that the diffuse traversal time exceeds the Heisenberg time. One may apply this to classical wave propagation in a medium with resonant scatterers. Due to resonant scattering the transport velocity v_E can be small [23], i.e. much smaller than c_0. As a result, the diffusion constant $D = \frac{1}{d}v_E\ell^*$ will be small, so that the propagation time L^2/D is relatively long. One is tempted to conclude that g becomes small and localization may occur. Yet, resonant scattering has no direct relation with localization. Indeed, one can show that the Heisenberg time is *equally* enlarged, because the density of states $\rho(E)$ per unit volume becomes large near resonant scattering [100, 101]. As a result, delay in resonant scattering does not affect the criterion for localization.

An important question is what happens to localization when absorption enters the problem. I will discuss some aspects of this question in Chapter 6. Here I want to define the Thouless dimensionless conductance g for an open sample with absorption. This problem has been investigated experimentally by Genack etal. [51, 102] for microwaves. According to random matrix theory, discussed in Chapter 2, and without dissipation, the probability distribution of the total transmission T_a depends functionally on the Thouless parameter $\langle g \rangle = N\ell^*/L$, which is essentially the second moment of the total transmission: $(\Delta T_a)^2/\langle T_a \rangle^2 \sim 1/g$, c.f. Eq. (24). In the presence

of absorption, this relation can still serve to define a new Thouless parameter g_a. It turns out empirically that the fluctuations in both T_a and T_{ab} are still correctly modeled by the random matrix results provided that the conductance is taken as [51].

$$\frac{(\Delta T_a)^2}{\langle T_a \rangle^2} \sim \frac{1}{g_a}. \tag{58}$$

In the presence of absorption this conductance no longer relates to the real conductance. If $g_a < 1$, one enters a regime with anomalous fluctuations, due to localization. In the last Chapter we shall discuss the role of absorption on localization. At the time of writing there is no elegant Thouless argument of the kind outlined above, why the conductance g_a emerges as a universal parameter in the presence of absorption.

4.2.1. Conductance and Boundary Conditions

The Thouless relation (56) implies a deep connection between conductance and typical energy scales. The ambiguity of the Thouless parameter g is that it addresses energy-levels in an open system. On the other hand, an open system cannot have exact energy eigenvalues, because the spectrum is absolute continuous.

Indeed, the original argument by Thouless was slightly different. Thouless actually considered a *closed* system with some boundary conditions. He defined δE as the sensitivity of the eigenstate to a change of boundary conditions. In the localized regime, the wave functions decay exponentially with a length much smaller than the system size so that a modification on the boundary will in general not change a lot. This leads again to criterion (57).

A simple but instructive model exists in which this Thouless argument can be made more precise. Following Akkermans and Montambaux [103] we consider a ring of length L enclosing a magnetic flux Φ. The advantage of this model is that exact eigenvalues exist because the systems is closed. In spite of this, the "Ohmic" conductance G can be defined as the factor of proportionality between induced Lenz current I and the time-derivative of the flux $d\Phi/dt$: $I = G\,d\Phi/dt$, as if the system would be open. The Hamiltonian for the ring is

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}), \tag{59}$$

where $V(\mathbf{r})$ is some random potential in the ring. Periodic boundary conditions are chosen for the wave function: $\psi(x + L) = \psi(x) \exp(i\phi)$, where x is the coordinate along the ring.

The first question is how the eigenvalues change for small phase. The equation for $u(\mathbf{r}) = \psi(\mathbf{r}) \exp(-i\phi x/L)$ reads,

$$H = \frac{(\mathbf{p} - \phi\,\hat{\mathbf{x}}/L)^2}{2m} + V(\mathbf{r}), \tag{60}$$

From this equation it follows that the phase shift ϕ can be associated with an Aharonov-Bohm flux Φ, and it not just a mathematical trick on the boundary. For small phase, the "perturbing" potential is $\delta V = -\phi p_x/mL + \phi^2/2mL^2$. Second-order perturbation theory immediately gives,

$$\frac{\partial E_n(\phi)}{\partial \phi^2} = \frac{1}{2mL^2} + \frac{1}{4m^2L^2} \sum_{k \neq n} \frac{|\langle \psi_n | p_x | \psi_k \rangle|^2}{E_n - E_k}. \tag{61}$$

The ensemble average of this quantity, divided by the mean level spacing can be defined as the dimensionless conductance,

$$g \equiv \langle N(E) \rangle \left\langle \frac{\partial E_n(\phi = 0)}{\partial \phi^2} \right\rangle. \tag{62}$$

This definition is a more precise variant of definition (56). It has been proven by Akkermans and Montamboux [103] that in the metallic regime $g \gg 1$, definition (62) coincides *rigorously* with the Ohmic conductance G defined above. In the localized regime this relation breaks down.

4.3. SCALING THEORY

After the work of Thouless, an important development was the scaling theory of localization, a phenomenological theory put forward in 1979 by the so-called "gang of four" [13], and inspired by the theory of phase transitions. For this reason, the onset of localization is often called an "electronic" phase transition [7]. This theory is believed to be somewhat dated as it considers only the ensemble-average of the conductance, and disregards fluctuations. Nevertheless, it has been of historical importance, as it set a trend for all work that followed. In addition to the clear, pioneering paper by the gang of four, good descriptions of the ideas behind scaling theory can be found in the works by Thouless [1], Ramakrishnan [2] Vollhardt and Wölfle [7], and recently Ping Sheng [64]. For that reason, I will restrict myself to a brief outline.

The basic question addressed by the scaling theory is the dependence of the conductance (transmission, summed over all channels in and out) on the volume of the system. Classically, that is without considering interference, the total conductance of the system depends on the transverse transverse A and its length L according to

$$g \sim \sigma(E)\frac{A}{L} = \rho(E)D(E)L^{d-2} \, . \tag{63}$$

The second equality assumes a disordered system with equal size in all directions (a cube or sphere). In three dimensions this means that the conductance increases when the system size finally increases, whenever the diffusion constant is nonzero. In one dimension, g would decrease. In the previous section we mentioned that if the conductance becomes small enough, one enters the localized regime.

From the previous argument it follows that increasing the systems size will in principle not help to approach localization. Only in one dimension this will be the case, and indeed all states are localized in one dimension. Quasi one-dimensional systems, i.e three dimensional systems where $0 < A < \ell^2$ is kept fixed, are subject to the same conclusion that all states are localized if the system is sufficiently long, consistent with criterion (20). In three dimensions increasing the systems size will only help if one is already in the localized regime. In that case

$$g \sim \exp\left(-\frac{L}{\xi}\right) \, . \tag{64}$$

It looks like if the size dependence of the conductance, its derivative in particular, depends on the conductance itself. The scaling theory of localization asserts the existence of a length ξ, depending on the amount of disorder in the system but *independent* of the sample size L, such that the dimensionless conductance depends only on the ratio ξ/L,

$$g = F\left(\frac{\xi}{L}\right) \, . \tag{65}$$

This equation implies the existence of a one-parameter scaling function

$$\beta(g) = \frac{\mathrm{d}\log g}{\mathrm{d}\log L} \, , \tag{66}$$

which depends *only* on the dimensionless conductance g. This identifies g as the universal scaling parameter. The scaling ansatz asserts $\beta(g)$ to be a smooth and continuous function of g. From Eqs. (63) and (64) one anticipates the limits,

$$\beta(g) = \begin{cases} d - 2 - a/g & g \gg g_c \\ \log g + \text{constant} & g \ll g_c \end{cases} \tag{67}$$

The term a/g is the weak localization correction that can be calculated from perturbation theory. In three dimensions the scaling ansatz together with both asymptotic limits leads us to the conclusion that some critical point g_c exists that obeys $\beta(g_c) = 0$. This point is identified as the critical value for the conductance in the Thouless criterion for localization (57). For $g > g_c$ one has $\beta > 0$ meaning that the conductance increases with sample size. In that case ξ is interpreted as a correlation length. If $g < g_c$ the conductance decreases with sample size, and ξ is identified as the localization length. These notions are consistent with the existence of extended and localized states respectively. Hence, the scaling theory provides an additional - though rather intuitive - support for Thouless' criterion. I want to stress that the assertion of a *continuous* scaling function immediately excludes a discontinuous conductivity near g_c as claimed by Mott. It is however, not a proof since continuity has been asserted, and not established. The derivative

$$\left.\frac{\mathrm{d}\beta}{\mathrm{d}\log g}\right|_{g_c} = \frac{1}{\nu}, \tag{68}$$

can be seen to determine the critical exponents on both sides of the transition. Near the critical point, Eq. (68) implies that $\log g = \log g_c + (L/\xi)^{1/\nu}$. Only the choice $\xi \sim |\omega - \omega_c|^{-\nu}$ makes the conductance g analytic near the mobility edge ω_c. In the extended regime we expect that $D \sim |\omega - \omega_c|^s$ and relation (63) indicates that $\xi \sim |\omega - \omega_c|^{-s/(d-2)}$. One-parameter scaling imposes,

$$s = \nu(d-2). \tag{69}$$

This hyperscaling equation was first derived using an elegant scaling theory [104]. In particular, in 3D the exponents ν and s are equal. Fig. 9 shows the expected qualitative behavior of the scaling function $\beta(g)$ in different dimensions.

The fixed point g_c has one important consequence for the total transmission T_a. In three dimensions, the relation between (ensemble-averaged) conductance g and transmission T_a is $T \approx g/(kL)^2$, where $(kL)^2$ is proportional to the number of conducting modes. At the transition, $g = g_c$ is independent of the size of the sample. At the same time we expect $k \approx 1/\ell$. Hence,

$$T(L) \approx \left(\frac{\ell}{L}\right)^2, \tag{70}$$

a relation that has been claimed by two recent localization experiments, one with microwaves [44] and one with light [45].

One of the major shortcomings of the scaling theory is the absence of fluctuations. Actually, the conductance g figuring in it is the ensemble-averaged conductance $\langle g \rangle$. It has been pointed out in Chapter 2 that many long-range fluctuations in transmission are also functions of $\langle g \rangle$. As such, $\langle g \rangle$ is still believed to be the most important parameter. Nevertheless, a theory of the type discussed in this section, but incorporating mesoscopic fluctuations, does not exist.
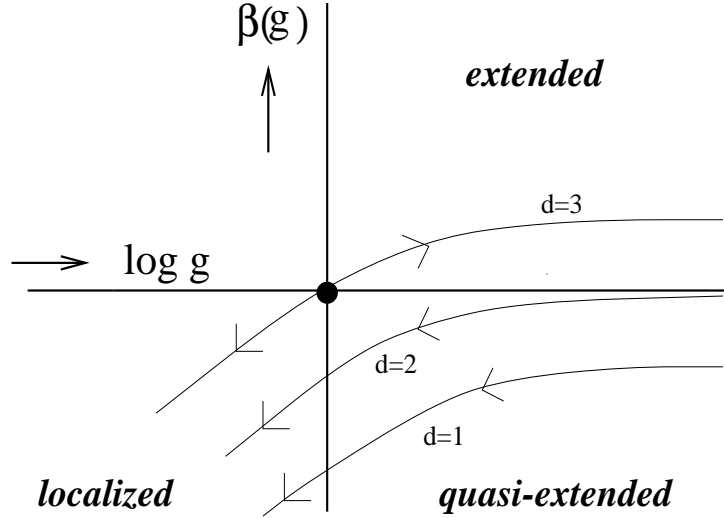
Figure 9. The scaling function $\beta(g)$, in dimensions $d = 1$, 2 and 3. In dimensions $d > 2$ a critical point exists defined as a zero of β. The three regimes have been indicated. The left upper sheet is forbidden by Thouless' criterion. The arrows denote the sense of scaling when the system size increases. A quasi 1D system (recall Figure 2) would scale here like an ordinary 1D system.

4.3.1. *Scaling Theory for the Anderson Model*

It is important to check the one-parameter scaling ansatz in situations that allow an exact numerical solution. Finite-size scaling in the 2D and 3D Anderson model, given by Eq. (35) has been extensively studied by MacKinnon and Kramer [105].

Consider the Anderson model on a 2D cubic lattice with $M \times N$ sites. We shall impose periodic boundary conditions in the M direction, so that we are studying a cylinder of length N and radius M. For $i = 1, \cdots N$ the M-dimensional vector wave function Ψ_i can be defined as $\Psi_i \equiv (\psi_{i1}, \psi_{i2}, \cdots \psi_{iM})$, and obeys the equation,

$$V \left(\Psi_{i+1} + \Psi_{i+1} \right) + \mathbf{E}_i \cdot \Psi_i = E \Psi_i . \tag{71}$$

This equation looks like the 1D Anderson model (36). The diagonal $M \times M$ matrix $\mathbf{E}_i$ contains the random site energies ε_{ij}. As the longitudinal size $N \to \infty$ we expect all states to be localized along the cylinder, with localization length ξ_M. As M increases, this system approaches the 2D Anderson model. If ξ_M converges to a final value ξ_∞ we can conclude that the 2D model is localized with localization length ξ_∞. If ξ_M stays larger than M, states must be extended in the 2D Anderson model.

Like has been done in ansatz (65) we can postulate the existence of a length ξ, independent of M such that

$$\Lambda \equiv \frac{\xi_M}{M} = F \left(\frac{\xi}{M} \right) . \tag{72}$$

Similarly a scaling function $\beta(\Lambda)$ can be defined, following Eq.(67). Numerically one now faces the considerable job to find, for each value of the disorder parameter W/V, an appropriate ξ such that all calculations fall on the same curve, which then determines b the still unknown function $F(x)$ in Eq. (72). In Fig. 10 we show the results of MacKinnon and Kramer [105].
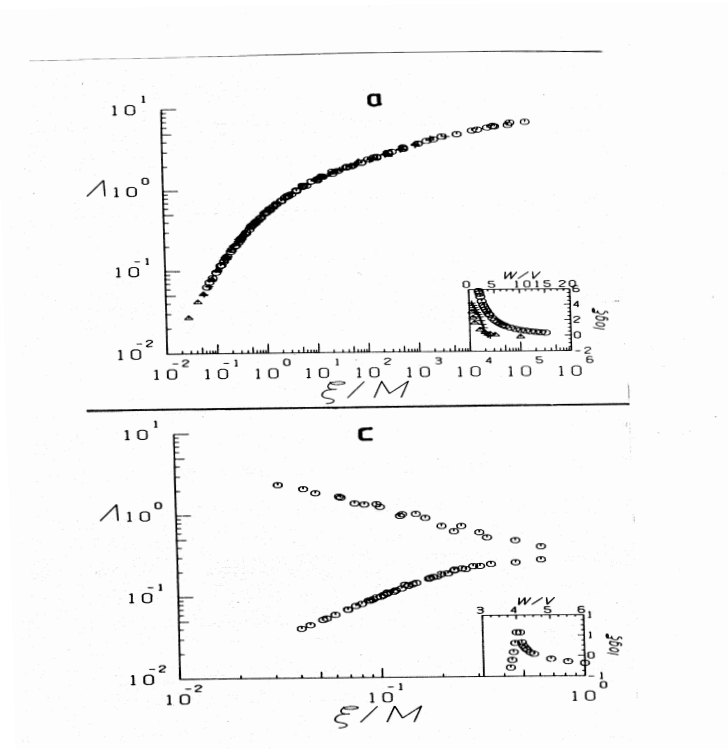
Figure 10. The scaling function $F(\xi/M)$ defined in Eq. (65) for the 2D Anderson model on a cylinder. Top: Diagonal disorder, which gives completely localized behavior. Bottom: Off-diagonal (sympletic) disorder, which shows a mobility edge at $W/V \approx 4$. Taken from MacKinnon [62].

The top figure shows the 2D Anderson model with diagonal disorder, the bottom figure shows an "off-diagonal" variant used to model spin-orbit scattering (in random matrix theory this would correspond to "$\beta = 4$"). Both graphs confirm the scaling hypothesis (72) beautifully. We can infer that diagonal disorder results in $\Lambda(M)$ that strictly decreases as M increases. Thus, $\beta(\Lambda) < 0$ so that according to the previous section all states are localized. The off-diagonal disorder seems to have a mobility edge even in 2D, as apparent from the upper branch where $\beta(\Lambda) > 0$.

5. Self-Consistent Theory of Localization

Localization is the phenomenon that wave functions become localized due to disorder. As may have become clear from the previous Chapters, one of the most important issues has been how localization influences the transport of waves through disordered media. One would like to identify a microscopic mechanism that induces localization by destroying explicitly the familiar picture of diffusion. This problem is not going to be easy. Classical transport theory predicts the transport to be diffuse, and any small perturbation caused by interference will modify this picture quantitatively, but not qualitatively: diffusion will still persist but may be characterized by a different diffusion constant. A theory beyond perturbation is needed. We mentioned already in Chapter 2 that random matrix theory is capable of describing the diffuse regime all the way to the localized regime in a nonperturbative way and may be called a "good" theory. A "good"

theory is characterized by a rigorous underlying principle, perhaps worked out in an approximate manner but in principle generally valid, and - last but not least - consistent with great principles such as the scaling theory of localization. The selfconsistent theory of localization is also a "good" theory, but like random matrix theory is suffers from several restrictions. An additional advantage of the selfconsistent theory is that it gives relatively simple formulas, from which the physics is apparent. Its disadvantage is that, unlike random matrix theory, it is a theory for the average intensity and diffusion constant only.

5.1. RECIPROCITY PRINCIPLE AND WEAK LOCALIZATION

The fundamental element of the selfconsistent theory of localization is the reciprocity principle. As outlined below, classical transport theory can never be a "good" theory for waves because it doesn't obey the reciprocity principle. I will first outline how transport theory is set up and how reciprocity comes is.

The basic observable in a transport experiment is the specific intensity (in reflection or transmission) which is proportional the complex field amplitude squared: $I(\mathbf{r}, t) = |\psi(\mathbf{r}, t)|^2$. Transport theory tries to formulate a relation between the ensemble-average of incident intensity and an outgoing intensity. More generally one can try to connect space-time correlation functions of incident and outgoing field. Since we consider media with linear response, the relation must be,

$$\langle \psi(\mathbf{r}_1, t_1) \psi^*(\mathbf{r}_3, t_3)) \rangle = \int_2 \int_4 \Gamma(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3, \mathbf{r}_4, t_1, t_2, t_3, t_4) \langle \psi(\mathbf{r}_2, t_2) \psi^*(\mathbf{r}_4, t_4)) \rangle , \qquad (73)$$

where the integrals are done over time and space. This relation identifies the vertex $\Gamma(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3, \mathbf{r}_4, t_i)$ as the fundamental object independent of incident field. Let us for simplicity assume monochromatic waves, so that the time-dependence $\exp(-i\omega t_i)$ becomes trivial. The reciprocity relation for Γ interchanges detector and source, i.e. incoming and outgoing position vectors,

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3, \mathbf{r}_4) = \Gamma(\mathbf{r}_2, \mathbf{r}_1; \mathbf{r}_4, \mathbf{r}_3) = \Gamma(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_4, \mathbf{r}_3) . \qquad (74)$$

The first equality is a reciprocity relation for intensity, the second one concerns the complex field field only.

Transport theory aims to find a simple expression for Γ in terms of microscopic properties of the scatterers and their statistics. A multiple scattering sequence can be identified in terms of the so-called irreducible vertex $U(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3, \mathbf{r}_4)$, defined by the matrix relation,

$$\Gamma = U + U \cdot G \times G^* \cdot \Gamma \equiv U + R , \qquad (75)$$

in which G is the averaged amplitude Green's function. R is a new object that contains only geometrical multiplications of the vertex U. Sometimes it is convenient to think as U in terms of a super single scattering, and R in terms of super multiple scattering. This is the closest one can get to the familiar multiple scattering picture without making any approximations. The adjective "super" is used to remind that U represents "one" scattering from a still complicated object that cannot be disentangled further without giving up the ensemble averaging. As such, U can be called a collision operator. Boltzmann transport theory replaces U by the scattering from one scatterer. This low-density approximation turns R into a genuine incoherent multiple scattering series, which obeys the familiar radiative transport equation. This equation disregards interference in multiple scattering.

It is important to realize that a price has to be paid by disentangling the exact solution into two operators R and U. The object R alone does not obey the reciprocity relation (74).

The ones that have to be added to R to restore reciprocity are contained in a set C defined by interchanging bottom indices of R,

$$C(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3, \mathbf{r}_4) = R(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_4, \mathbf{r}_3) \ . \tag{76}$$

This set is not part of the set of "reducible" events contained in R, but is part of U. The object C is the rigorous mathematical definition for the set of "most-crossed diagrams", a jargon that is used to refer to diagrams that generate all kinds of interference effects in multiple scattering, such as coherent backscattering and weak localization. The collision operator U can now be decomposed into

$$U(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3, \mathbf{r}_4) = C(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3, \mathbf{r}_4) + S(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3, \mathbf{r}_4). \tag{77}$$

By construction, the subset S is closed under the operation carried out Eq. (76), just like the total set Γ. Contrary to C it also exists classically. Physically it contains "loop" events that finally return to where they came from. In the theory of coherent backscattering, R describes "background" and C the "coherent peak". It is relation (76) that guarantees a coherent backscattering peak with an enhancement factor of approximately 2. It is only approximately 2, because the vertex S does not give a peak, and contributes to background [106]. In Figure 11 we show typical diagrammatic representations of the vertices R, S and C, evaluated in wave number space.

We now come to the basic point of the selfconsistent theory for localization. The vertex $\Gamma = R + S + C$ finally determines the ensemble-averaged intensity. The reciprocity relation (74) states that the solution of Eq. (75) is necessarily a selfconsistent equation for the vertex R, since C is basically the same object as R and serves as input of Eq. (75). In particular, the Boltzmann approximation disregards C and thus violates the reciprocity principle! The reciprocity principle forbids the existence a transport equation with separated microscopic input and mesoscopic output.

5.2. MICROSCOPIC THEORY FOR DIFFUSION CONSTANT

The next issue is how the conclusion of the previous section influences wave transport, the diffusion constant in particular. The first thing to notice is that energy conservation "proves" that wave propagation can be described by a diffusion constant at long time and length scales (mathematicians would not agree with this statement). More precisely, the object $\Gamma((\mathbf{r}_1 \approx \mathbf{r}_3; \mathbf{r}_2 \approx \mathbf{r}_4) = \Gamma(\mathbf{r}_{12})$ obeys a diffusion equation. The diffusion constant can be calculated from the following exact Kubo formula [107],

$$\rho(\omega) D(\omega) = \frac{1}{3\pi} \int \frac{d\mathbf{p}}{(2\pi)^3} \left[p^2 |G(\omega, p)|^2 \gamma(\omega, p) - \frac{\partial \mathrm{Re}\, G(\omega, p)}{\partial p^2} \right] , \tag{78}$$

where the object $\gamma(\omega, p)$ must satisfy the equation

$$\gamma(\omega, p) = 1 + \int \frac{d\mathbf{p}'}{(2\pi)^3} \frac{\mathbf{p} \cdot \mathbf{p}'}{p^2} |G(\omega, p')|^2 U_{\mathbf{p}\mathbf{p}'}(\omega)\, \gamma(\omega, p') \ . \tag{79}$$

In these equations, ω is the frequency (we consider classical waves here, with vacuum velocity 1) and $\mathbf{p}$ the momentum. The vertex $U_{\mathbf{p}\mathbf{p}'}(\omega)$ and the Green's function $G(\omega, p)$ have been introduced formally in the previous section and are here presented as matrix elements in wavenumber space, using the convention of Figure 11; $\rho(\omega)$ is the density of states per unit volume. We have
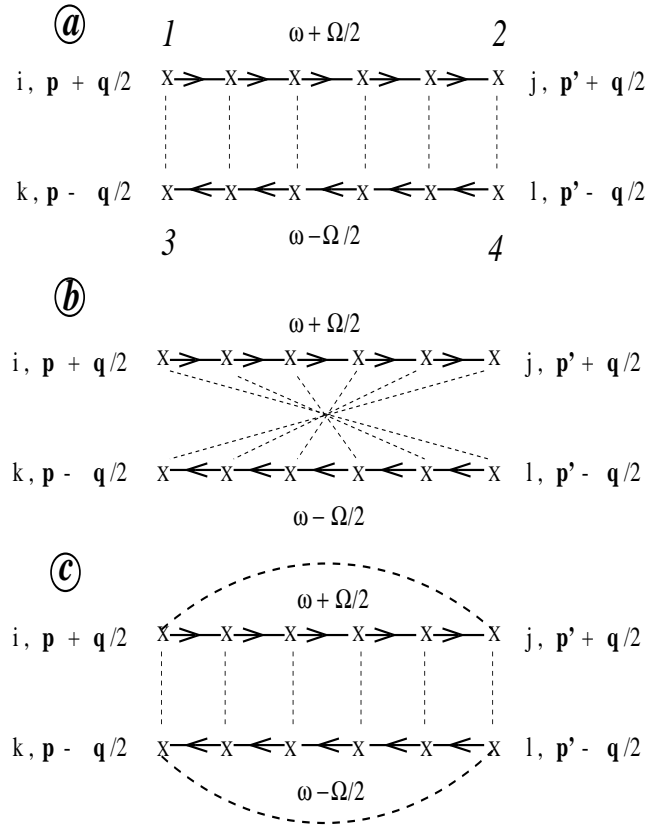
Figure 11. Typical diagrammatic representations of two correlated events $\langle \psi(a)\psi(b)^* \rangle$. By convention, the top line denotes $\psi(a)$ and bottom line the complex conjugate $\psi^*(b)$ (more precisely the propagation at energy $\omega - \Omega/2 - i0$, which explains why it propagates in the opposite direction); a, b denote paths with different frequencies, different wave numbers and different polarizations at in- and output, as indicated in the graphs; Crosses denote scattering events. Dashed lines connect identical scatterers. The numbers 1,2,3 and 4 denote the convention in Eqs. (73) and (74). A). Typical contribution to the reducible vertex R. The present graph represents a totally incoherent event (a "ladder" diagram), where the phase cancels. B). Typical contribution to the vertex C ("most-crossed" diagram), obtained by time-reversing the lower path of the vertex R. C.) Typical contribution to the vertex S which is here an incoherent path with equal start and end ("loop").

mentioned that U and Γ are related by reciprocity. As a result, U obeys a diffusion equation too, *with the same diffusion constant.* In this way one can see that Eqs. (78) and (79) are in fact coupled equations for the diffusion constant $D(\omega)$.

This selfconsistency was first pointed out by Götze [108] and worked out in detail by Vollhardt and Wölfle [7, 15, 16]. In the diffuse regime the Fourier transform of $R(\mathbf{r})$ is (see Fig. 11),

$$R_{\mathbf{p}\mathbf{p}'}(\Omega, \mathbf{q}) \approx \frac{4\pi}{\ell^2} \frac{1}{-i\Omega + Dq^2} . \tag{80}$$

The relation (76) can be reformulated in wave number space, and the object $C_{\mathbf{p}\mathbf{p}'}$ takes the form

$$C_{\mathbf{p}\mathbf{p}'}(\Omega, \mathbf{q}) \approx \frac{4\pi}{\ell^2} \frac{1}{-i\Omega + D(\mathbf{p} + \mathbf{p}')^2} \tag{81}$$

Inserting $U_{\mathbf{pp'}} = S + C_{\mathbf{pp'}}$ into Eq. (79) and making some intuitive approximations, one arrives at, for $\Omega = 0$,

$$\frac{1}{D(\omega)} = \frac{1}{D_0(\omega)} + \frac{C_d}{\rho(\omega)\ell} \int d^d\mathbf{q} \frac{1}{D(\omega)q^2} \,. \tag{82}$$

This equation is called the selfconsistent equation for the diffusion coefficient; C_d is a known constant that depends on dimensionality. The first term is the diffusion constant free from interference. In the form it has been written down, it applies to any dimension. It can be adapted for anisotropy or inelastic scattering or absorption [7]. Its straightforward dynamical generalization, i.e. $\Omega \neq 0$ will be discussed in the next Chapter.

The wavenumber integral in Eq. (82) suffers from a number of divergencies that should be considered in more detail. In dimensions $d \leq 2$ it diverges at small wave numbers. This corresponds physically to long wave trajectories, for which the approximations leading to Eq. (82) should apply. This catastrophe implies that interference effects inhibit classical diffusion in $d \leq 2$ completely, as argued many times already in the present paper. In dimensions $d \geq 2$ the integral diverges at large q, indicating that some of our approximations start to break down at small lengths. We expect this length to be equal to the mean free path, a conclusion that emerges from a detailed analysis [93]. In dimensions $d > 2$ we put $q_{\max} \approx 1/\ell$ so that

$$D(\omega) = D_0(\omega) \left(1 - \frac{C_d}{\rho(\omega)\ell^{d-1}} \right) \,. \tag{83}$$

This equation predicts localization - in the sense of criterion (1) - to occur when $\rho(\omega)\ell^{d-1} = C_d$, which agrees with the generalized Ioffe-Regel criterion (54) formulated earlier, realizing that the density of states scales as $\rho(\omega) \sim k^{d-1}$. A precise consideration of C_d demonstrates that $(k\ell)^{d-1} \sim (d-2)/d$ at the mobility edge.

The second interesting aspect of Eq. (82) is the relative easy to investigate the effect of a finite open medium in an approximate way. In that case we anticipate trajectories longer than the system size L to be absent. Hence a minimum value $q_{\min} \approx 1/L$ may be considered. In 3D, the selfconsistent equation can be rewritten as ,

$$\frac{1}{D(\omega, L)} \left(1 - \frac{C_3}{\rho(\omega)\ell^2} \right) = \frac{1}{D_0(\omega)} \left(1 - \frac{C_3'}{g(\omega, L)} \right) \,. \tag{84}$$

Here $g(\omega, L) \sim \rho(\omega)D(\omega, L)L$ denotes the dimensionless conductance, defined earlier in Eq. (63). The advantage of this form is that both the Ioffe-Regel criterion (55) and the Thouless criterion (56) appear. Because the system is finite, the diffusion constant never really vanishes but at the mobility edge of the infinite medium, described by the Ioffe-Regel criterion, the dimensionless conductance defined in Eq. (56) equals some constant of order unity. As a result, the diffusion constant D scales as $1/L$, a conclusion that also emerged from the scaling theory of localization. The selfconsistent equation for $g(\omega, L)$ satifies the scaling ansatz (65) with a correlation length that diverges if $k\ell \approx 1$. For more details I refer to the excellent review paper by Vollhardt and Wölfle [7].

In the localized regime and for an infinite system the diffusion constant D vanishes, its role being taken over by the localization length ξ. It is now necessary to consider the dependence of the diffusion constant on the hydrodynamic frequency Ω. We set $D(\Omega) = -i\Omega\xi^2(\omega)$, so that Eq. (80) for the reducible vertex R takes the form,

$$R(\Omega, \mathbf{q}) \approx \frac{4\pi}{-i\Omega\ell^2} \frac{1}{1 + \xi^2(\omega)q^2} \,. \tag{85}$$

In real space and in the time domain this functional dependence describes a stationary, and exponentially small (ensemble averaged) intensity correlation, as specified earlier in Eq. (9). The selfconsistent equation for the localization length is easily seen to be,

$$\frac{1}{\xi^2(\omega)} = \frac{C_d}{\rho(\omega)} \int_{q<\ell} \mathrm{d}^d\mathbf{q} \, \frac{1}{1 + q^2\xi^2(\omega)} \, . \tag{86}$$

The solution of this equation heavily depend on dimensionality. In one dimension, $\xi \approx \ell$, a not too surprising conclusion in view of Chapter 3. In 2D, all states are still localized with localization length $\xi \approx \ell \exp(\rho\ell)$, consistent with the relation (46) obtained for the 2D Anderson model. In three dimensions the localization length diverges at the mobility edge according to,

$$\xi(\omega) \sim \frac{\ell}{(k\ell)_c - k\ell} \, , \tag{87}$$

i.e. with a critical exponent $\nu = 1$. In the selfconsistent theory, the case of four dimensions is a critical upper dimension. For $d > 4$ it can be inferred that

$$\xi(\omega) \sim \frac{\ell}{[(k\ell)_c - k\ell]^{1/2}} \tag{88}$$

i.e. a critical exponent $\nu = \frac{1}{2}$ *independent* of the dimensionality. Note that such an upper critical dimensionality was not obtained for the Cayley tree.

Figure 6 of Chapter 3 shows a comparison of the phase diagram as predicted by the self consistent equation and the exact numerical solution for the 3D Anderson model. It can be seen that the agreement is excellent. This agreement suggests that localization is for a great deal a catastrophe in weak localization (because that is the physics that goes in), a statement on which not everybody agrees. One of the problems with the selfconsistent theory is that it predicts typical mean-field critical exponents around the transition, whereas numerical simulations (of the Anderson model) indicate the critical exponents to be different. As a result, the selfconsistent equation is believed to break down very close to the transition. Also when considering magnetic fields, in which case the basic reciprocity principle (76) underlying the selfconsistent theory breaks down, the selfconsistent theory disagrees with the now generally accepted picture that wave localization is not destroyed - at most modified - by broken time-reversal symmetry.

6. Localization in the Time Domain

Some time-dependent features of localization have been briefly addressed in Chapter 2. The time-evolution of the center-of mass $\mathbf{r}(t)$ of a wave packet plays an important role and was seen to be involved already in the very definition of the phenomenon. Time dependent features of wave propagation are - by Fourier transformation - related to frequency correlations.

Many more dynamical features of localization have been studied. For electronic systems the basic dynamic observable is the dynamic AC (electronic) conductivity, i.e. the conductivity of electrons driven by an oscillating electric field $E(\Omega) \sim \exp(-i\Omega t)$ implying that electronics mainly deals frequency formulations. In ultrasonics or seismology, observations are carried out directly in the time domain. In this Chapter we will have a closer look how localization is assumed to affect dynamical wave propagation. Time dependent wave localization in one-dimensional systems has been studied out loud by Sheng, Papanicolaou etal. , in particular in relation to its potential applications in seismology. Finally, time-dependent acoustic wave propagation has

been investigated numerically by Weaver etal. A dynamic version of random matrix theory is now in rapid development.

6.1. FREQUENCY CORRELATIONS IN 1D

Localization of acoustic waves in one dimension has been extensively studied by Sheng, Papanicolaou, White and Zhang [10, 109, 110, 111]. It may have applications to seismology in view of the layered structure of the Earth. The source initiating wave propagation, such as Earth quakes and dynamite explosions, are typically limited in time. Observations are therefore intrinsically carried out in the time-domain. The 1D solution is important because it provides a rigorous treament of time-dependent fluctuations in the localized regime, that may also be relevant to other domains, and perhaps even to higher dimensions.

Consider a semi-infinite layered medium. Different layers may have different size or different acoustic properties, such as different wave velocity and different density. The background velocity and density are given by v_0 and ρ_0. The wave equation reads

$$i\partial_t \mathbf{\Psi}(z,t) = \mathbf{H}(z)\cdot\mathbf{\Psi}(z,t) \,. \tag{89}$$

The wave function $\mathbf{\Psi}(z,t) = (p,v)$ contains pressure and velocity variations over the background. The "Hamiltonian" is given by

$$\mathbf{H}(z) = \begin{pmatrix} 0 & -K(z)p_z \\ \rho^{-1}(z)p_z & 0 \end{pmatrix} \tag{90}$$

which may be symmetrized by a change of variables. Both the elastic bulk modulus $K(z)$ and the density $\rho(z)$ are 1D random functions. The statement is that "almost" all states are exponentially localized. If we consider monochromatic solutions of this wave equation for the semi-infinite stack, the localization implies that the the reflection coefficient has modulus one for "almost all" frequencies ω, i.e. $R(\omega) = \exp(i\psi)$. In this 1D model the absolute value is not subject to fluctuations, unless of course we allow some leaks at the bottom. The phase, however, fluctuates and one may expect that $\langle R \rangle = 0$. Information may be retrieved from the spectral correlation function,

$$U(\omega_1,\omega_2) = \langle R(\omega_1)R^*(\omega_2)\rangle \,. \tag{91}$$

This correlation function can be deduced from the fluctuations of the pressure field $p(t,z=0)$ at the top. Consider an initial pulse at $z=0$ whose frequency distribution is described by $F(\omega)$. The pressure field $p(t,z=0)$ is given by,

$$p(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \exp(i\omega t)F(\omega)R(\omega) \,. \tag{92}$$

Carring out some Fourier transformations we find the simple relation,

$$U\left(\omega - \frac{1}{2}\Omega, \omega + \frac{1}{2}\Omega\right) = \frac{1}{|F(\omega)|^2}\int_{-\infty}^{\infty} \mathrm{d}\tau \exp(-i\Omega\tau)\int_{-\infty}^{\infty} \mathrm{d}t \exp(-i\omega t)\langle p(\tau+t)p(t)\rangle \,. \tag{93}$$

We assumed the typical value for Ω to be much smaller than ω, so that $F(\omega)$ can be considered constant over the band Ω. Correlations over a band width Ω typically probe wave paths less than v_0/Ω.

Sheng etal. [10] have calculated the spectral correlation function U in a special but relevant limit. This limit applies when $a \ll \lambda \ll \xi(\omega)$, i.e. in case of separation of the length scales a (the thickness of the layers), the wave length $\lambda \sim v_0/\omega$ and the 1D localization length $\xi(\omega)$. This limit has been studied by various authors [10, 112]. It is found that,

$$U\left(\omega - \frac{1}{2}\Omega, \omega + \frac{1}{2}\Omega\right) \approx \int_0^\infty \mathrm{d}s \frac{s}{s + iY} \exp(-s) \,. \tag{94}$$

The correlation function depends on one parameter $Y = \Omega\xi(\omega)/v_0$ only . An important conclusion is that the time-dependent correlation function give direct access to the frequency dependence of the localization length, since the dephasing frequency Ω is typically equal to $v/\xi(\omega)$, i.e. the inverse time to travel one localization length. This also confirms that $|\Omega| \ll |\omega|$ as has been assumed earlier. At low frequencies $\omega \to 0$ the localization length $\xi \sim \omega^{-2}$ is large and dephasing occurs rapidly.

Equation (94) can be transformed into one for the power spectrum $S(\omega, \tau)$ of fluctuations with frequency ω in a time window τ. The result is

$$S(\omega, \tau) = \int_{-\infty}^\infty \mathrm{dt} \, \exp(-i\omega t) \, \langle p(\tau + t)p(t) \rangle = |F(\omega)|^2 \times \frac{1}{|\tau|} \frac{\chi}{(1 + \chi)^2} \,. \tag{95}$$

where $\chi = |\tau| v_0/\xi(\omega)$. The fluctuations at frequency ω are suppressed once the time window of observation τ exceeds the time to traverse one localization length. A very useful conclusion, which I think, is worth investigating in higher dimensional or quasi-one dimensional systems.

6.2. TIME-DEPENDENT TIGHT BINDING MODELS

Ultrasonic localization experiments have initiated the study of a time-dependent variant of the Anderson model (35),

$$\partial_t^2 \psi_n + \varepsilon_n \psi_n - V \sum_m{}' \psi_m = F_n(t) \,. \tag{96}$$

Here $\{\varepsilon_n\}$ are again identical, independently distributed stochastic variables, and V is a hopping element from a site to a nearest neighbour. They have a uniform distribution between $-\frac{1}{2}W$ and $\frac{1}{2}W$. The second-order time derivative suggests that we are dealing with classical waves although one should not forget that light and ultrasound would have a time dependent interaction of the form $(1 + \varepsilon_n)\partial_t^2 \psi_n$. The function $F_n(t)$ denotes a source term and rapidly decays to zero for large times and at sites far away from the source.

Equation (96) has been studied numerically in 2D by Weaver [113]. For the 2D Anderson model all states are known to be localized. It is found that beyond a certain time the *ensemble-averaged* wave energy takes a stationary profile that is decays exponentially away from the source. This is consistent with localization and in particular with criterion (3). For all values of the microscopic parameter W/V, and for all distances from the source, the time-evolution towards the stationary state seems to be a universal function, empirically given by,

$$|\psi_n(t)|^2 \sim \exp\left[\frac{-r_n}{\xi} - \left(\frac{r_n^{2.46}}{4D_p t \xi^{0.46}}\right)^{0.76}\right] \,, \tag{97}$$

where ξ is the localization length and D_p is a quantity with the dimension of diffusivity. So far, no microscopic theory is able to explain this numerical result. The exponent $2.46 > 2$ indicates that

the spread of energy is already sub-diffuse $\langle \mathbf{r}^2(t) \rangle \sim t^{0.81}$ well before the wave packet becomes localized.

An important recent development for *ab initio* time-dependent studies is the wave automaton. This is an efficient numerical tool that considers random S-matrices on a large lattice. These S-matrices control the scattering at each site. If they are taken random, but unitary, they model the random hopping of a wave packet from one site to the other, in much the same way as the Anderson model. In fact, it can be shown that the wave automaton mimics a time-dependent version of the Anderson model with diagonal and second-nearest neighbour coupling [114].

The appearance of a subdiffusive regime, eventually leading to a bounded $\langle \mathbf{x}^2(t) \rangle$ at large times has been confirmed by Vanneste, Sebbah and Sornette in numerical studies using the 2D wave automaton [114]. They report a behavior $\langle \mathbf{r}^2(t) \rangle \sim t^{0.90}$, not far from the subdiffuse law reported by Weaver for the Anderson model. The wave automaton exhibits amazingly long times, up to 10^6 times the intersite travel time, to become saturated due to the onset of localization. To my knowledge, the wave automaton has never been studied in the case of 3D. Another interesting aspect may be to investigate broken time-reversal symmetry, which according to random matrix theory should modify localization.

6.3. AC CONDUCTIVITY & TIME-DEPENDENT DIFFUSION CONSTANT

The selfconsistent theory of localization can be straightforwardly generalized for the AC diffusion constant $D(\omega, \Omega)$. Here ω is the microscopic cycle frequency, for electrons typically equal to k_F/v_F, i.e. the "Fermi" frequency and for classical waves simply the central frequency of the pulse; Ω is the carrier frequency of the envelope, determining the dynamics of the wave packet. In electronic language, a conductance or diffusivity dependent on Ω can be thought of as a parallel capacitance.

In general one has $|\Omega| \ll |\omega|$. One might anticipate that for small Ω the diffusion pole still dominates, i.e. $R(\Omega, \mathbf{q}) \sim 1/[-i\Omega + D(\Omega)q^2]$, so that Eq. (82) generalizes to

$$\frac{1}{D(\omega, \Omega)} = \frac{1}{D_0(\omega)} + \frac{C_d}{\rho(\omega)\ell} \int d^d\mathbf{q} \frac{1}{-i\Omega + D(\omega, \Omega)q^2}. \tag{98}$$

Like always with frequency response functions, $R(\Omega, \mathbf{q})$ must be an analytic function in the (physical) sheet $\text{Im}\,\Omega > 0$, since at times $t < 0$ there was no pulse at all. The diffusion pole is determined by the analytic continuation of this function in the sheet $\text{Im}\,\Omega < 0$.

I will focus on the time dependent behavior of an infinite 3D random medium. In the extended regime of can identify three regimes,

$$D(\Omega) = \begin{cases} D_0 & \Omega\tau > 1 \\ (i\Omega\tau)^{1/3} & (D(0)/D_0)^3 < \Omega\tau < 1 \\ D(0) & \Omega\tau < (D(0)/D_0)^3 \end{cases} \tag{99}$$

with $\tau = 3D_0/\ell$ the mean free time. This qualitative behavior of $D(\Omega)$ agrees with the time-dependent scaling by Berkovits and Kaveh [115]. Exactly at the mobility edge one finds $D(\Omega) \approx D_0(i\Omega\tau)^{1/3}$, in agreement with the elegant scaling theory first set up by Wegner [104]. In real space and in the time domain such AC diffusion would yield a time tail of the kind,

$$R(t \to \infty, r) \sim \frac{1}{rt^{2/3}}, \tag{100}$$

rather than the familiar diffuse result $R(t \to \infty, r) \sim 1/(D_0 t)^{3/2}$ for an infinite diffuse medium. The time-dependent reflection coefficient $R(t)$ of a semi-infinite medium at the mobility edge was estimated by Berkovits and Kaveh [115] to vary as $R(t) \sim 1/t^{4/3}$. The time dependent transmission through a slab of length L at the mobility edge is estimated to be [116]

$$T(t \to \infty) \sim \frac{1}{t^{2/3}} \exp\left(-\gamma t^{2/3}/L^2\right) , \tag{101}$$

to be compared to the outcome $T(t) \sim \exp(-Dt/L^2)$ of the conventional diffusion equation. It would be fascinating to guide time-dependent localization experiments in this spirit. Also 3D seismic localization may possibly be deduced from anomalous time tails in this way.

6.3.1. *Role of Dissipation*
So far we have discussed localization without considering dissipation. As a result, Hamiltonians are symmetric operators and S-matrices are unitary. Experiments with classical waves always deal with some sort of dissipation. It is important to think the concept of localization over again in the presence of absorption. Absorption is easiest to understand in the time domain.

The definitions of localization given in section 2.1 all break down for one reason or another when absorption comes into play. In the case of light, absorption is described by a conductivity $\sigma(\mathbf{r})$ (at optical frequencies). The wave equation for the electrical field reads,

$$\varepsilon(\mathbf{r})\partial_t^2 \mathbf{E}(\mathbf{r}, t) + \nabla \times \nabla \times \mathbf{E}(\mathbf{r}, t) = -\sigma(\mathbf{r})\partial_t \mathbf{E}(\mathbf{r}, t) . \tag{102}$$

I shall assume both dielectric constant $\varepsilon(\mathbf{r})$ and conductivity $\sigma(\mathbf{r})$ to be independent of frequency. The simplest situation occurs when the absorption time $\tau_a = \varepsilon(\mathbf{r})/\sigma(\mathbf{r})$ is independent of $\mathbf{r}$, i.e. the absorption is homogeneous. In that case, one has, on top of the time evolution due to multiple scattering, an extra exponential decay in time,

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r}, t)\, e^{-t/2\tau_a} . \tag{103}$$

The field $E_0(\mathbf{r}, t)$ satisfies the wave equation,

$$\varepsilon(\mathbf{r})\partial_t^2 \mathbf{E}_0(\mathbf{r}, t) - \frac{\varepsilon(\mathbf{r})}{4\tau_a^2}\mathbf{E}_0(\mathbf{r}, t) + \nabla \times \nabla \times \mathbf{E}_0(\mathbf{r}, t) = 0 . \tag{104}$$

This is essentially the wave equation for an electric field without absorption. The second, new term can be seen to be very small since for small absorption $\omega\tau_a \gg 1$. Therefore, Eq. (104) can serve to define localization (of light) in the presence of dissipation, using the result of Chapter 1. In particular, the center-of-mass $\mathbf{r}_0(t)$ is introduced as,

$$\mathbf{r}_0^2(t) \equiv \langle \mathbf{E}_0(t)|\,\varepsilon(\mathbf{r})\,\mathbf{r}^2\,|\mathbf{E}_0(t)\rangle = e^{t/\tau_a}\,\langle \mathbf{E}(t)|\,\varepsilon(\mathbf{r})\,\mathbf{r}^2\,|\mathbf{E}(t)\rangle . \tag{105}$$

A "diffusion constant" can then be introduced as,

$$D_a = \frac{1}{d}\lim_{t\to\infty}\left\langle \frac{\mathbf{r}_0^2}{t}\right\rangle = \frac{1}{d}\lim_{t\to\infty} e^{t/\tau_a}\left\langle \frac{\mathbf{r}^2}{t}\right\rangle . \tag{106}$$

Even without making the assumption of homogeneous absorption, one might *speculate* about the existence of some minimal time τ_a such that Eqs. (105) and (106) are finite. I don't know of
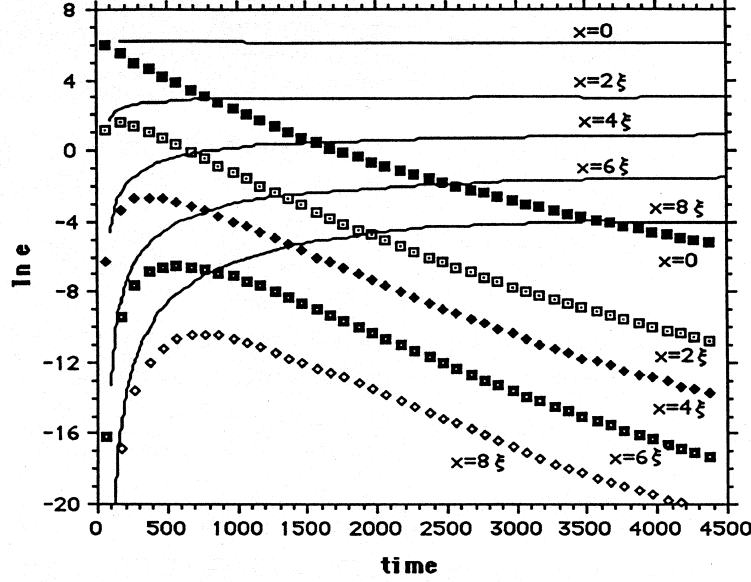
Figure 12. The evolution of the energy density in a viscous 2D Anderson model with $W/V = 11$, for different distances from the source. Solid: undamped; symbols denote damped system. Taken from Weaver [113], with kind permission from the author.

any exact results for this problem. Localization in infinite absorbing media can now be defined as a bounded $\mathbf{r}_0^2(t)$ in time (for "almost all realizations"). More weakly one can impose $D_a = 0$.

It is straightforward to apply the selfconsistent theory of localization, discussed in the previous Chapter, to Eq. (104), as done by Josefin [117]. The conclusion is almost trivial: this equation does not suffer from absorption, and therfore, the final equation for the diffusion constant D_a will also be free from absorption, so that all localization phenomena will remain unaltered, except for the "trivial" exponent $\exp(-t/\tau_a)$. Numerical experiments carried out by Weaver [113, 118] for the 2D Anderson model with absorption, confirm this. In Figure 12 I show numerical results by Weaver [118] for the energy density $\psi(n,t)^2$ on the 2D Anderson model with diagonal disorder and a viscous term $\sigma_n \partial_t \psi(n,t)$ in much the same way as in Eq. (102), with the difference that also σ_n is chosen random ("diagonal random absorption"). The graph shows that except for an overall time decay of the kind $\exp(-t/\tau_a)$, the (ensemble-averaged) wave packet remains exponentially localized in space, as discussed in Chapter 2. The physical idea behind is that absorption only kills the amplitude but not the phase. In this respect absorption of classical waves is fundamentally different from inelastic scattering processes in the solid state. Inelastic scattering does not affect the amplitude but destroys the phase. The electron is not destroyed but is reset. Genack etal [102] multiplied their microwave data in transmission with the same factor $\exp(t/\tau_a)$ to arrive again at the conclusion that absorption is an almost trivial complication in the experiment that can be restored in the computer.

Despite these many strong arguments I believe that statements like "localization is not modified by absorption" and "absorption is a trivial aspect of localization" are too strong. When τ_a is chosen too small in Eqs. (105) and (106) one is actually putting gain into the medium, and the limits tend to infinity. At the time of writing it is unclear to me how this statement relates

to the known and perhaps unexpected result that "gain" actually lowers the localization length in just the same way as absorption does. This was first shown for one-dimensional media [76] and recently for quasi one-dimensional media [119].

Secondly, the problem is that definition (106) does not always describe the measurement. Average intensities and field correlation functions are always described by the vertex R defined in Chapter 5. In the presence of absorption, the expression for R changes into,

$$R(\Omega, \mathbf{q}) \sim \frac{1}{-i\Omega + D(\Omega)\mathbf{q}^2 + 1/\tau_a} \,. \tag{107}$$

It can easily be checked that $D(\Omega = -i\tau_a) = D_a$, defined in Eq. (106). But stationary measurements, among which the DC conducivity featuring in the scaling theory of localization, are properties for $\Omega = 0$. In that case, the selfconsistent equation becomes,

$$\frac{1}{D(\omega)} = \frac{1}{D_0(\omega)} + \frac{C_d}{\rho(\omega)\ell} \int \mathrm{d}^d \mathbf{q} \frac{1}{D(\omega)q^2 + 1/\tau_a} \,. \tag{108}$$

We remark that the argument of reciprocity leading to this selfconsistent equation for diffusion is not violated in the presence of absorption. It can easily be shown from Eq. (108) that the absorption term destroys the mobility edge and $D > 0$ for all values of the mean free path, and in any dimension. Near the mobility edge of a nonabsorbing 3D system Eq. (108) predicts a highly nontrivial dependence of the diffusion constant on the absorption time τ_a, $D \sim 1/\tau_a^{1/3}$, which is the counterpart of Eqs. (99). It is well known that the length $L_a \equiv \sqrt{D\tau_a}$ describes the exponential fall-off of the ensemble-averaged transmission [47]. Thus, near $k\ell = 1$,

$$T(L) \sim \exp(-L/L_a) \sim \exp(-\gamma L/\tau_a^{1/3}) \,. \tag{109}$$

This relation agrees with the "theory of white paint" by Anderson [9].

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