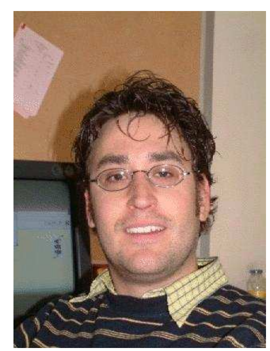
Leaky Repeated Interaction Quantum Systems *

Alain JOYE

* Joint work with



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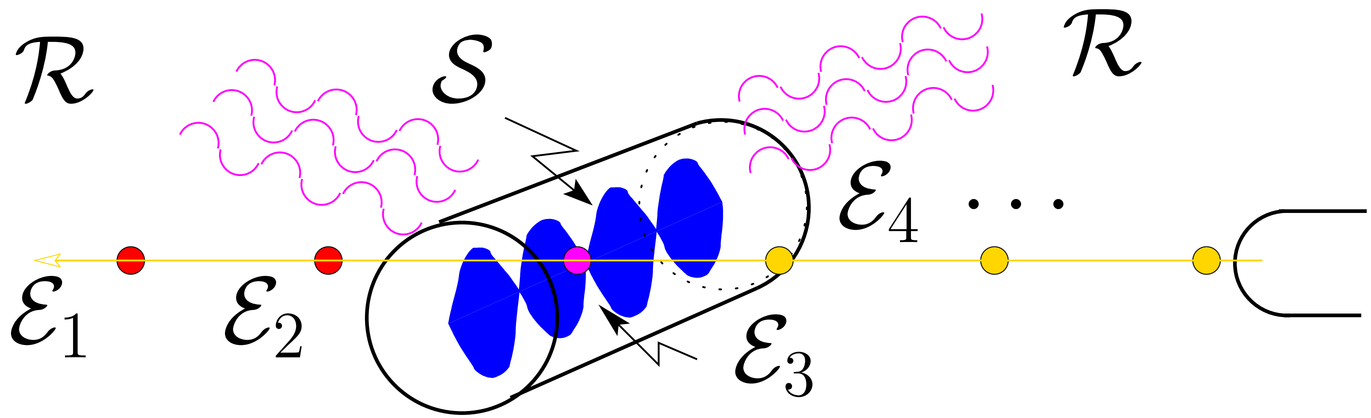
Motivation

One-atom maser Walther et al ’85, Haroche et al ’92



- $\mathcal{S}$: one mode of E-M field in a cavity
- $\mathcal{E}_k$: atom $\#k$ interacting with the mode
- $\mathcal{C}$: sequence of atoms passing through the cavity
- $\mathcal{R}$: environment responsible for losses

Ideal RIQS used as simple models Vogel et al 93, Wellens et al 00, BJM 06, Bruneau Pillet 09

Random RIQS to model fluctuations BJM 08

Leaky RIQS to account for losses

The Formal Model

Quantum system $\mathcal{S}$:

- Finite dimensional system, driven by Hamiltonian $H_{\mathcal{S}}$ on $\mathfrak{H}_{\mathcal{S}}$, s.t. $\sigma(H_{\mathcal{S}}) = \{e_1, \cdots, e_d\}$.

Chain $\mathcal{C}$ of identical quantum sub-systems $\mathcal{E}_k \equiv \mathcal{E}$, $k = 1, 2, \cdots$:

$$\mathcal{C} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 + \cdots$$

- Each $\mathcal{E}_k$ is driven by the Hamiltonian $H_{\mathcal{E}\,k} \equiv H_{\mathcal{E}}$ on $\mathfrak{H}_{\mathcal{E}_k} \equiv \mathfrak{H}_{\mathcal{E}}$, $\dim \mathfrak{H}_{\mathcal{E}} \leq \infty$
- The chain $\mathcal{C}$ is driven by $H_{\mathcal{C}} \equiv H_{\mathcal{E}\,1} + H_{\mathcal{E}\,2} + \cdots$ on $\mathfrak{H}_{\mathcal{C}} \equiv \mathfrak{H}_{\mathcal{E}_1} \otimes \mathfrak{H}_{\mathcal{E}_2} \otimes \cdots$, with $[H_{\mathcal{E}\,j}, H_{\mathcal{E}\,k}] = 0, \ \ \forall j, k.$

Fermionic reservoir $\mathcal{R}$:

- ∞ -ly extended gas of indep. fermions at temperature β , driven by $"H_{\mathcal{R}}"$ on $"\mathfrak{H}_{\mathcal{R}}"$.

The Formal Model

Complete system $\mathcal{S} + \mathcal{R} + \mathcal{C}$

- Formal Hilbert space $\mathfrak{H}_\mathcal{S} \otimes$ "$\mathfrak{H}_\mathcal{R}$" $\otimes \mathfrak{H}_\mathcal{E}$

Interaction $\mathcal{S} - \mathcal{C}$

- $W_{\mathcal{S}\mathcal{E}}$ operator on $\mathfrak{H}_\mathcal{S} \otimes \mathfrak{H}_{\mathcal{E}_k}$, $k = 1, 2, \cdots$.

Interaction $\mathcal{S} - \mathcal{R}$

- $W_{\mathcal{S}\mathcal{R}}$ operator on $\mathfrak{H}_\mathcal{S} \otimes$ "$\mathfrak{H}_\mathcal{R}$".

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Interaction $\mathcal{S} - \mathcal{R}$

- $W_{\mathcal{S}\mathcal{R}}$ operator on $\mathfrak{H}_\mathcal{S} \otimes {}''\mathfrak{H}_\mathcal{R}''$.

Evolution Let $\tau > 0$ be a duration, $\lambda = (\lambda_\mathcal{R}, \lambda_\mathcal{E}) \in \mathbb{R}^2$ be couplings

For $t = (m-1)\tau + s, \;\; 0 \le s < \tau$,

- $\mathcal{S}$, $\mathcal{R}$ and $\mathcal{E}_m$ are driven by $H_\mathcal{S} + {}''H_\mathcal{R}'' + H_\mathcal{E} + \lambda_\mathcal{R} W_{\mathcal{S}\mathcal{R}} + \lambda_\mathcal{E} W_{\mathcal{S}\mathcal{E}}$
- $\mathcal{E}_k$ evolve freely with $H_\mathcal{E}$, $\forall\, k \neq m$

Leaky Repeated Interactions Quantum Systems

Pictorially

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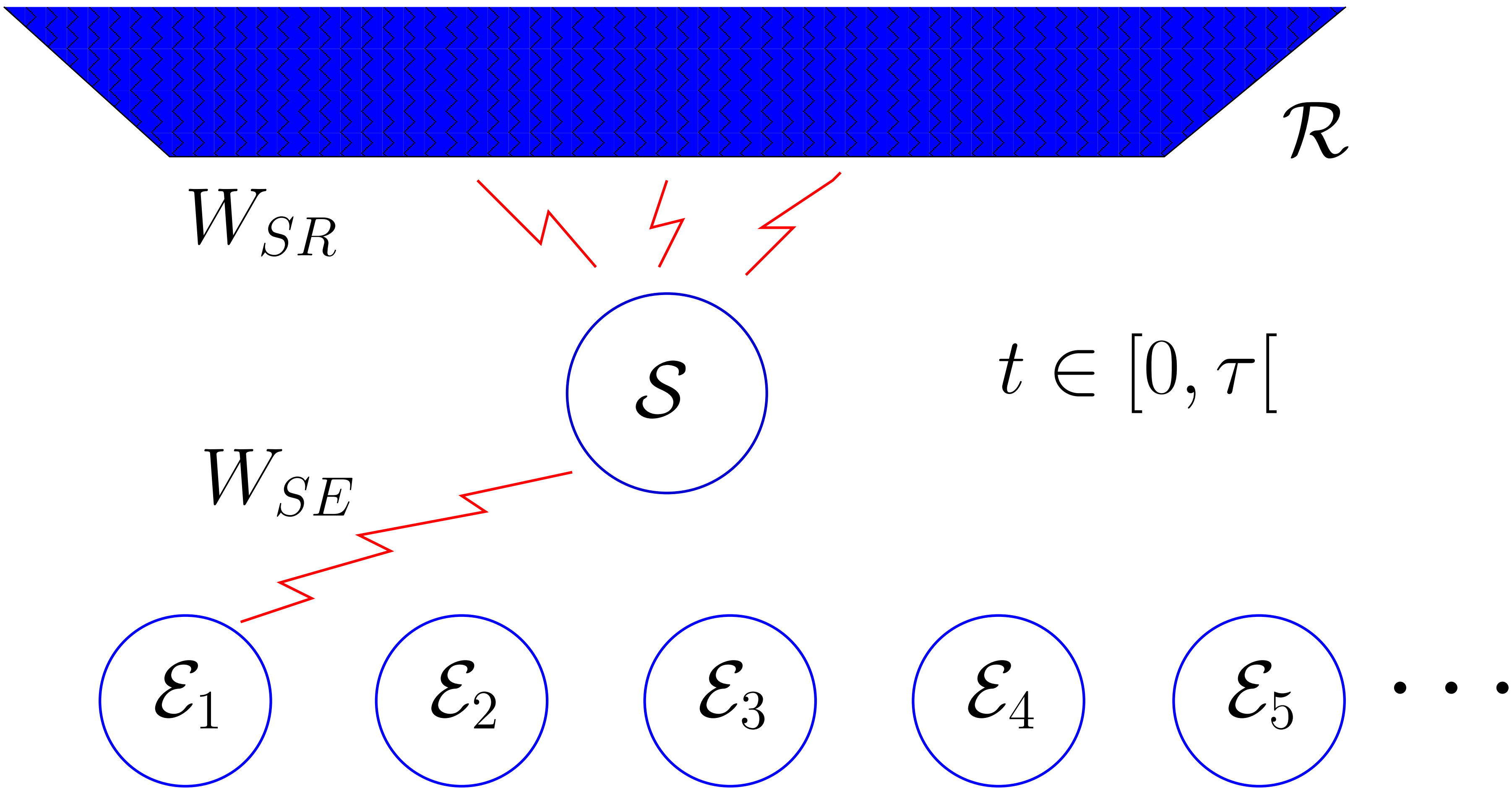
Pictorially



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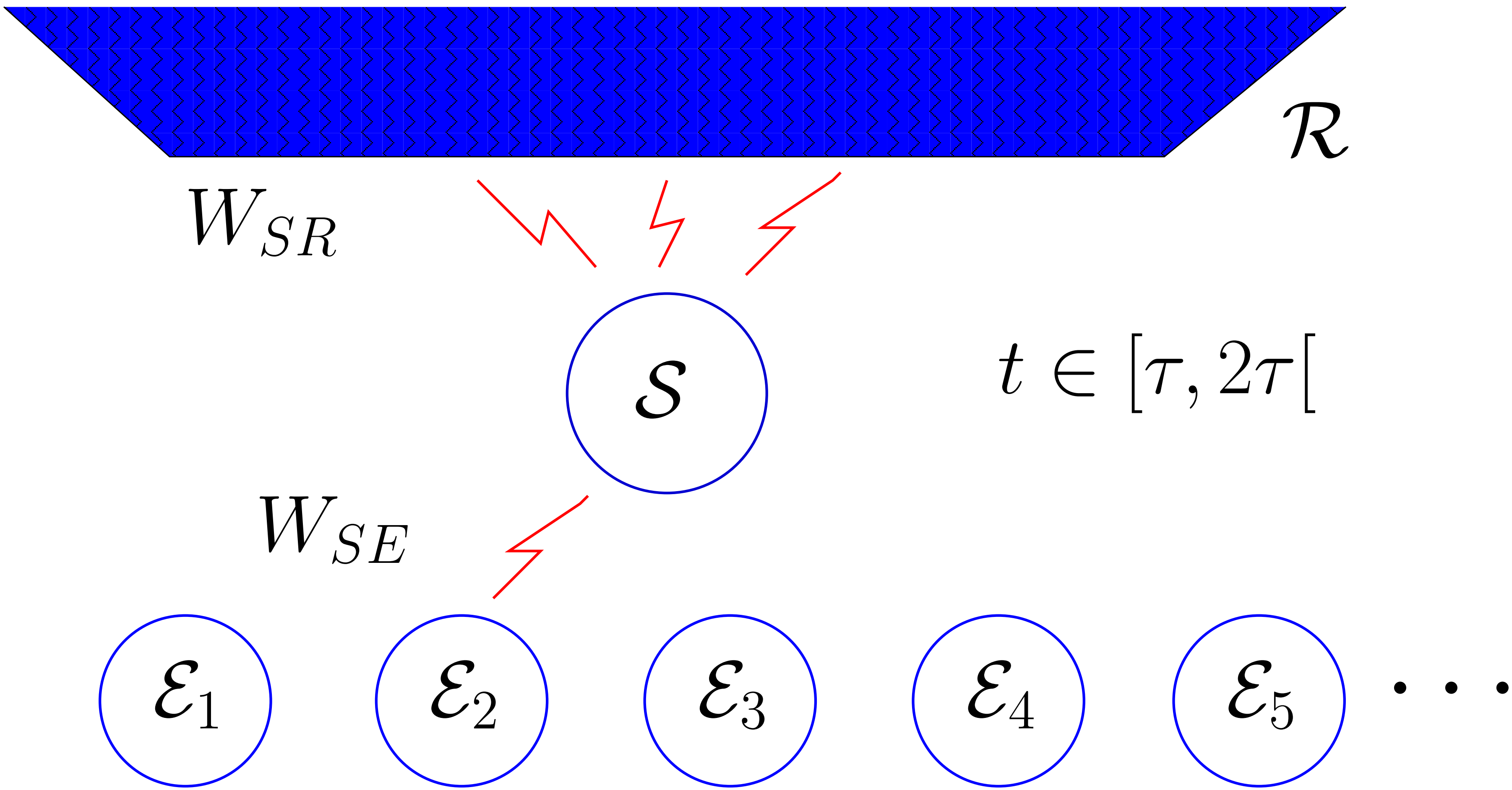

Leaky Repeated Interactions Quantum Systems

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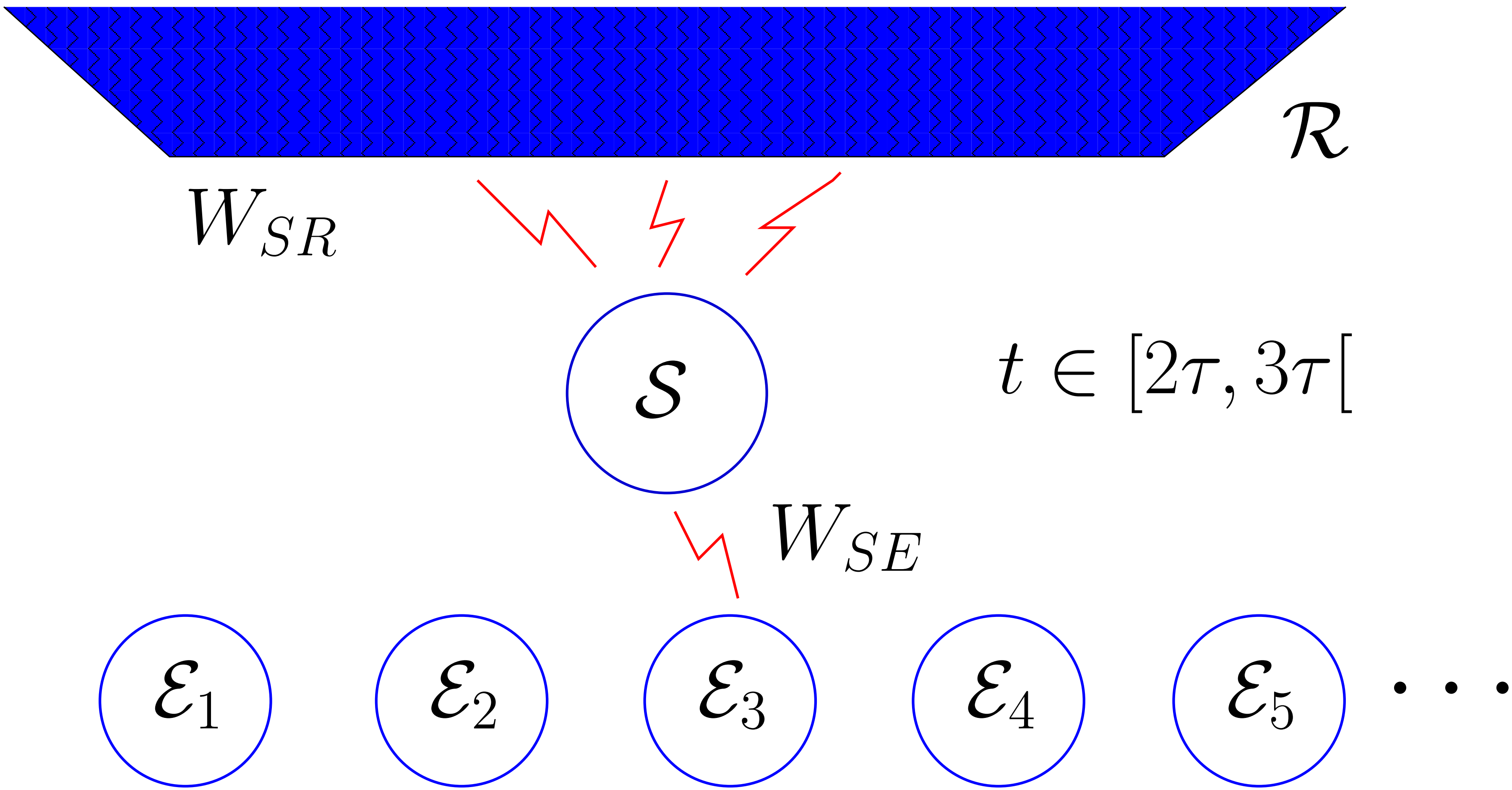

Leaky Repeated Interactions Quantum Systems

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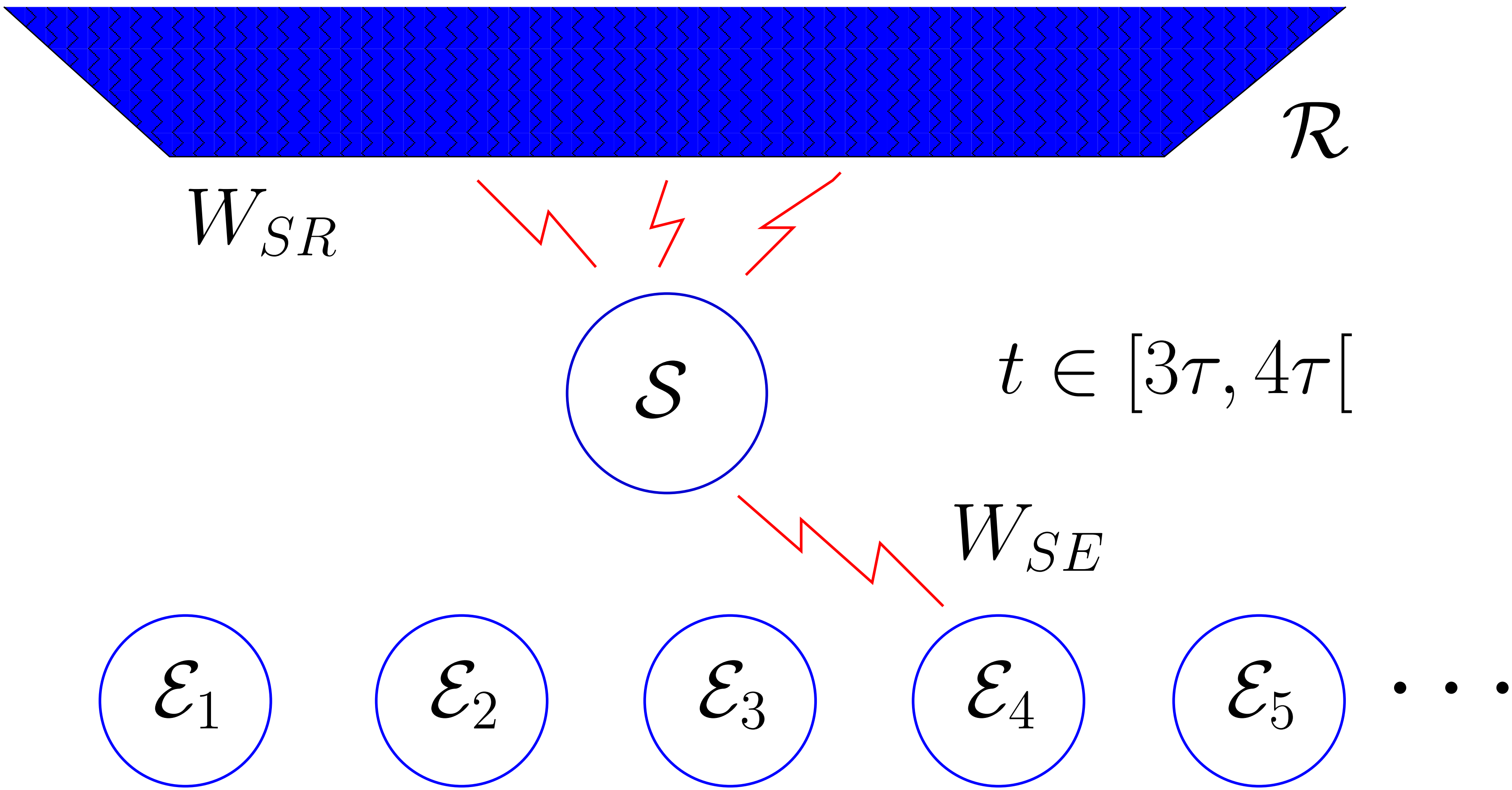

Questions

Large times asymptotics

Let $A = A_{SR} \otimes \mathbb{I}_{\mathcal{C}} \in \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes \text{"}\mathfrak{H}_{\mathcal{R}}\text{"} \otimes \mathfrak{H}_{\mathcal{C}})$ an observable acting on $\mathcal{S} + \mathcal{R}$

Let $\alpha^t(A)$ be its Heisenberg evolution, at time $t = m\tau$

Let $\rho : \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes \text{"}\mathfrak{H}_{\mathcal{R}}\text{"} \otimes \mathfrak{H}_{\mathcal{C}}) \to \mathbb{C}$ be a state (“density matrix”)

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Let $\rho : \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes "\mathfrak{H}_{\mathcal{R}}" \otimes \mathfrak{H}_{\mathcal{C}}) \to \mathbb{C}$ be a state (“density matrix”)

- Existence of $\lim_{m\to\infty} \rho \circ \alpha^{m\tau}(A) = \rho^+(A)$?

  Dependence of $\rho^+(A)$ on the coupling constants $\lambda = (\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})$?

- Exchanges between $\mathcal{R}$ and $\mathcal{C}$ through $\mathcal{S}$?

  Energy variations, Entropy production, 2[nd] law of thermodynamics ?

- Non-trivial examples ?

Remark :

If $\lambda_{\mathcal{R}} = 0$, then $\mathcal{S} + \mathcal{C} \Rightarrow$ convergence to a NESS Bruneau-J.-Merkli 06

If $\lambda_{\mathcal{E}} = 0$, then $\mathcal{S} + \mathcal{R} \Rightarrow$ return to equilibrium Jaksic-Pillet 96

GNS Representation

Density matrix on $\mathfrak{H}$ $\quad\longrightarrow\quad$ pure state on $\mathcal{H} = \mathfrak{H} \otimes \mathfrak{H}$:

$$
\begin{array}{lrcl}
\text{state} & \rho = \sum \lambda_j |\varphi_j\rangle\langle\varphi_j| & \longrightarrow & \Psi_\rho = \sum \sqrt{\lambda_j}\varphi_j \otimes \varphi_j \\
\text{observable} & A \in \mathcal{B}(\mathfrak{H}) & \longrightarrow & \Pi(A) = A \otimes \mathbb{I}_{\mathfrak{H}} \in \mathcal{B}(\mathcal{H}) \\
\text{so that} & \mathrm{Tr}_{\mathfrak{H}}(\rho A) & = & \mathrm{Tr}_{\mathcal{H}}(|\Psi_\rho\rangle\langle\Psi_\rho|\Pi(A))
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\end{array}
$$

Dynamics

$$
\begin{array}{rcl}
A \in \mathcal{B}(\mathfrak{H}) & \mapsto & \alpha^t(A) = e^{itH} A e^{-itH} \in \mathcal{B}(\mathfrak{H}) \\
\rho \in \mathcal{B}_1(\mathfrak{H}) & \mapsto & e^{-itH} \rho e^{itH} \in \mathcal{B}_1(\mathfrak{H})
\end{array}
$$

Liouville operator

Given ρ invariant, $\exists$ a unique self-adjoint L on $\mathcal{H} = \mathfrak{H} \otimes \mathfrak{H}$ s.t.

$$
\left\{
\begin{array}{lcl}
\Pi(\alpha^t(A)) & = & e^{itL} \Pi(A) e^{-itL} \in \mathcal{H} \\
L\Psi_\rho & = & 0
\end{array}
\right.
$$

Simple setup

$$
L = H \otimes \mathbb{I}_{\mathfrak{H}} - \mathbb{I}_{\mathfrak{H}} \otimes H
$$

Fermi Heat Bath

Temperature β^{-1}

Ingredients: $\tilde{\mathfrak{h}} = L^2(\mathbb{R}^3, dk^3)$ one particle Hilbert space

one particle Hamiltonian $\tilde{h}$ s.t.

$(\tilde{h}\tilde{f})(k) = k^2\tilde{f}(k)$, $k \in \mathbb{R}^3$, $\forall \tilde{f} \in \tilde{\mathfrak{h}} = L^2(\mathbb{R}^3, dk^3)$

Hamiltonian $d\Gamma_-(\tilde{h})$ on Fock sp. $\Gamma_-(\tilde{\mathfrak{h}}) = \oplus_{n=0}^{\infty}\Gamma_-^n(\tilde{\mathfrak{h}})$

$a(\tilde{g})$, $a^*(\tilde{g})$ annih. and creat. op's on $\Gamma_-(\tilde{\mathfrak{h}})$, $\tilde{g} \in \tilde{\mathfrak{h}}$

Thermal state: ω_β characterized by 2 pts functions

$\omega_\beta(a^*(\tilde{g})a(\tilde{f})) = \langle \tilde{f} | (1 + e^{\beta\tilde{h}})^{-1}\tilde{g}\rangle$ and

$$\omega_\beta(a^*(\tilde{g}_n)\cdots a^*(\tilde{g}_1)a(\tilde{f}_1)\cdots a(\tilde{f}_n)) = \det(\omega_\beta(a^*(\tilde{g}_i)a(\tilde{f}_j)))$$

GNS for Fermi Bath

Jaksic-Pillet Gluing 02:

Change of variables: $k^2 \to s \in \mathbb{R}^+$, and $L^2(S^2, d\sigma) = \mathfrak{G}$

Enlarged Hilbert space "$L^2(\mathbb{R}^+, \mathfrak{G}) + L^2(\mathbb{R}^+, \mathfrak{G}) = L^2(\mathbb{R}, \mathfrak{G})$"

i.e. $\mathcal{H}_{\mathcal{R}} = \Gamma_-(\mathfrak{h})$, $\mathfrak{h} = L^2(\mathbb{R}, \mathfrak{G})$

Liouvillean $L_{\mathcal{R}} = d\Gamma(h)$, with h s.t.

$(hf)(s) = sf(s)$, $s \in \mathbb{R}$, $\forall f \in \mathfrak{h} = L^2(\mathbb{R}, \mathfrak{G})$

Creat., annih. op's $a^*(g_\beta)$, $a(g_\beta)$, where $g_\beta \leftrightarrow \tilde{g}$ via

$$g_\beta(s) = (e^{-\beta s} + 1)^{-1/2} g(s)\,,\; g(s) = \begin{cases} \tilde{g}(s) & \text{if } s \geq 0 \\ \bar{\tilde{g}}(-s) & \text{if } s < 0. \end{cases}$$

Equilibrium State $|\Psi_R\rangle\langle\Psi_{\mathcal{R}}|$, $\Psi_{\mathcal{R}}$ vacuum of $\Gamma_-(\mathfrak{h})$

Formalization

After GNS (writing A for $\Pi(A)$)

- Hilbert spaces $\mathcal{H}_\mathcal{S}$, $\mathcal{H}_\mathcal{R}$, $\mathcal{H}_{\mathcal{E}_k}$, and $\mathcal{H}_\mathcal{C} = \mathcal{H}_{\mathcal{E}_1} \otimes \mathcal{H}_{\mathcal{E}_2} \otimes \mathcal{H}_{\mathcal{E}_3} \otimes \cdots$
- Algebras of observables $\mathfrak{M}_\# \subset \mathcal{B}(\mathcal{H}_\#)$, $\# = \mathcal{S}, \mathcal{R}, \mathcal{E}, \mathcal{C}$
- States on $\mathcal{S}, \mathcal{R}, \mathcal{E}, \mathcal{C}$ are density matrices on $\mathcal{H}_\mathcal{S}$, $\mathcal{H}_\mathcal{R}$, $\mathcal{H}_\mathcal{E}$, $\mathcal{H}_\mathcal{C}$
- Evolution of observables $A_\mathcal{S} \mapsto \alpha^t_\mathcal{S}(A_\mathcal{S})$, $A_\mathcal{R} \mapsto \alpha^t_\mathcal{R}(A_\mathcal{R})$, $A_\mathcal{E} \mapsto \alpha^t_\mathcal{E}(A_\mathcal{E})$

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- Assumption: $\exists$ invariant states (cyclic and separating)

$\Psi_\mathcal{S} \in \mathcal{H}_\mathcal{S}$, $\Psi_\mathcal{R} \in \mathcal{H}_\mathcal{R}$ and $\Psi_\mathcal{E} \in \mathcal{H}_\mathcal{E}$ s.t.

$$\alpha^t_\#(A_\#) = e^{itL_\#} A_\# e^{-itL_\#}, \quad \text{and} \quad L_\# \Psi_\# = 0, \qquad \text{where} \quad \# = \mathcal{S}, \mathcal{R} \text{ and } \mathcal{E}$$

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$\alpha^t_\#(A_\#) = e^{itL_\#} A_\# e^{-itL_\#}$, and $L_\# \Psi_\# = 0$, where $\# = \mathcal{S}, \mathcal{R}$ and $\mathcal{E}$

System $\mathcal{S} + \mathcal{R} + \mathcal{C}$

On $\mathcal{H} = \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{R} \otimes \mathcal{H}_\mathcal{C}$, driven by $L_{\text{free}} = L_\mathcal{S} + L_\mathcal{R} + \sum_k L_{\mathcal{E}_k}$

Interactions

$V_{S\#} \in \mathfrak{M}_\mathcal{S} \otimes \mathfrak{M}_\#$, the GNS repres. of $W_{S\#}$, $\# = \mathcal{R}, \mathcal{E}$ + tech. hyp.

Dynamics

Repeated interaction Schrödinger dynamics

For any $m \in \mathbb{N}$, if $t = m\tau$ and $\psi \in \mathcal{H}$,

$$U(m)\psi := e^{-i\tilde{L}_m} e^{-i\tilde{L}_{m-1}} \cdots e^{-i\tilde{L}_1} \psi$$

where the generator for the duration τ is

$$\tilde{L}_m = \tau L_m + \tau \sum_{k \neq m} L_{\mathcal{E},k}$$

with

$$\begin{cases} L_m & = & L_{\mathcal{S}} + L_{\mathcal{R}} + L_{\mathcal{E}} + V_m & \text{on } \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}} \otimes \mathcal{H}_{\mathcal{E}_m} & \text{coupled} \\ V_m & = & \lambda_{\mathcal{R}} V_{\mathcal{S}\mathcal{R}} + \lambda_{\mathcal{E}} V_{\mathcal{S}\mathcal{E}} & & \\ L_{\mathcal{E},k} & = & L_{\mathcal{E}} & \text{on } \mathcal{H}_{\mathcal{E}_k} & \text{free} \end{cases}$$

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To be studied

Let $\varrho \in \mathcal{B}_1(\mathcal{H})$ be a state on $\mathcal{H}$ and $A_{\mathcal{S}\mathcal{R}} \in \mathfrak{M}$ an observable on $\mathcal{S} + \mathcal{R}$

$$m \mapsto \varrho(U^*(m) A_{\mathcal{S}\mathcal{R}} U(m)) \equiv \varrho(\alpha^{m\tau}(A_{\mathcal{S}\mathcal{R}})), \quad \text{as} \quad m \to \infty$$

Main Result

Theorem

Assume (A) and (FRG). For any state ϱ on $\mathcal{H}_{S\mathcal{R}} \otimes \mathcal{H}_{\mathcal{C}}$ and any analytic observable A_{SR}

$$\begin{aligned}
\lim_{n\to\infty} \varrho(\alpha^{\tau n}_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(A_{SR})) &= \langle \psi^*_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta_1) | A_{SR}(\theta_1)\Psi_{SR}\rangle \\
&\equiv \rho^+_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(A_{SR}),
\end{aligned}$$

where $\psi^*_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta_1) \in \mathcal{H}_{SR}$ is a “resonance vector” computable in perturbation theory.

Instantaneous Observables

$\exists$ Upgrade to more general observables !

A Simple Model

- $\mathcal{S}$ and $\mathcal{E}$ spins with e.v. $\{0, E_{\mathcal{S}}\}$, resp. $\{0, E_{\mathcal{E}}\}$
- $\mathcal{R}$ Fermi gas at $\beta_{\mathcal{R}}$, equil. state $\omega_{\beta_{\mathcal{R}}}$
- $W_{\mathcal{S}\mathcal{E}} = a_S \otimes a_E^* + a_S^* \otimes a_E$
- $\Psi_{\mathcal{S}}$ tracial, $\Psi_{\mathcal{E}} \simeq \omega_{\beta,\mathcal{E}} = e^{-\beta_{\mathcal{E}} H_{\mathcal{E}}}/Z_{\beta_{\mathcal{E}}}$
- $W_{\mathcal{S}\mathcal{R}} = \sigma_x \otimes (a_R^*(f) + a_R(f))$, $f \in L^2(\mathbb{R}^+, \mathfrak{G})$ “regular”.

Perturbation theory

1) If $\|f(\sqrt{E_{\mathcal{S}}})\| > 0$ and $\tau(E_{\mathcal{S}} - E_{\mathcal{E}}) \neq 2\pi\mathbb{Z}^*$, then (FGR) holds

2) The asymptotic state ω_+ is given by

$$\omega_+ = (\gamma\, \omega_{\beta_{\mathcal{R}},\mathcal{S}} + (1-\gamma)\, \omega_{\tilde{\beta}_{\mathcal{E}},\mathcal{S}}) \otimes \omega_{\beta_{\mathcal{R}}} + \mathcal{O}(\|(\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})\|)$$

with

$$\gamma = \frac{\lambda_{\mathcal{R}}^2 \frac{\pi}{2} \sqrt{E_{\mathcal{S}}}\|f(\sqrt{E_{\mathcal{S}}})\|^2}{\lambda_{\mathcal{R}}^2 \frac{\pi}{2} \sqrt{E_{\mathcal{S}}}\|f(\sqrt{E_{\mathcal{S}}})\|^2 + \lambda_{\mathcal{E}}^2 \tau \,\mathrm{sinc}^2(\tau(E_{\mathcal{S}} - E_{\mathcal{E}})/2)}, \qquad \tilde{\beta}_{\mathcal{E}} = \beta_{\mathcal{E}} \frac{E_{\mathcal{E}}}{E_{\mathcal{S}}}.$$

Energy

$\alpha^m(\tilde{L}_m)$ is (the GNS rep. of) "total energy" for times $t \in [(m-1)\tau, m\tau)$.

Variation between $(m+1)\tau$ and $m\tau$,

$$\Delta E^{\text{tot}}(m) \quad = \quad \alpha^{m+1}(\tilde{L}_{m+1}) - \alpha^m(\tilde{L}_m) = \alpha^m(V_{m+1} - V_m)$$

Similarly

$$\begin{aligned}
\Delta E^S(m) &= \alpha^{m+1}(L_{\mathcal{S}}) - \alpha^m(L_{\mathcal{S}}) \\
\Delta E^R(m) &= \alpha^{m+1}(L_{\mathcal{R}}) - \alpha^m(L_{\mathcal{R}}) \\
\Delta E^C(m) &= \alpha^{m+1}(L_{\mathcal{E}_{m+1}}) - \alpha^m(L_{\mathcal{E}_m})
\end{aligned}$$

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Similarly

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Asymptotic energy variation per unit time

$$dE_+^{\#} := \lim_{N\to\infty} \rho\left(\frac{\sum_{m=1}^N \Delta E^{\#}(m)}{N}\right)$$ exists under (A) and (FRG)

Property

$$dE_+^S = 0, \quad dE_+^{\mathrm{tot}} = dE_+^R + dE_+^C$$

Entropy production

Let $\Psi_{\mathcal{S}}$ and $\Psi_{\mathcal{E}}$ correspond to Gibbs states at temperatures $\beta_{\mathcal{S}}$ and $\beta_{\mathcal{E}}$

Relative entropy ϱ and ϱ_0 are states on $\mathfrak{M}$, generalization of

$$Ent(\varrho|\varrho_0) = \text{Tr}\,(\rho(\ln\rho - \ln\rho_0)) \geq 0$$ Araki '75

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Variation of relative entropy w.r.t. KMS states Jaksic, Pillet '03

Let $\varrho_0 \leftrightarrow \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{R}} \otimes \Psi_{\mathcal{C}}$, with $\Psi_{\mathcal{C}} = \Psi_{\mathcal{E}} \otimes \Psi_{\mathcal{E}} \otimes \cdots$ and ϱ be any state,

$$\Delta S(m) \quad := \quad Ent(\varrho \circ \alpha^m|\varrho_0) - Ent(\varrho|\varrho_0)$$

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$$\Delta S(m) \quad := \quad Ent(\varrho \circ \alpha^m|\varrho_0) - Ent(\varrho|\varrho_0)$$

Asymptotic entropy production rate

$$dS^+ \quad := \quad \lim_{N\to\infty} \frac{\Delta S(N)}{N}$$ exists and

$$dS^+ \quad = \quad \beta_{\mathcal{E}} dE_+^{\mathcal{C}} + \beta_{\mathcal{R}} dE_+^{\mathcal{R}}$$ 2^{nd} law

For the Model

With $\tilde{\beta}_{\mathcal{E}} = \beta_{\mathcal{E}} \frac{E_{\mathcal{E}}}{E_{\mathcal{S}}}$ and $\lambda = (\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})$ small

$$\begin{aligned}
dE_+^{\mathcal{C}} &= \kappa E_{\mathcal{E}} \left(e^{-\beta_{\mathcal{R}} E_{\mathcal{S}}} - e^{-\tilde{\beta}_{\mathcal{E}} E_{\mathcal{S}}} \right) + O(\lambda^3), \\
dE_+^{\mathcal{R}} &= \kappa E_{\mathcal{S}} \left(e^{-\tilde{\beta}_{\mathcal{E}} E_{\mathcal{S}}} - e^{-\beta_{\mathcal{R}} E_{\mathcal{S}}} \right) + O(\lambda^3), \\
dE_+^{\mathrm{tot}} &= \kappa (E_{\mathcal{E}} - E_{\mathcal{S}}) \left(e^{-\beta_{\mathcal{R}} E_{\mathcal{S}}} - e^{-\tilde{\beta}_{\mathcal{E}} E_{\mathcal{S}}} \right) + O(\lambda^3), \\
dS_+ &= \kappa (\tilde{\beta}_{\mathcal{E}} E_{\mathcal{S}} - \beta_{\mathcal{R}} E_{\mathcal{S}}) \left(e^{-\beta_{\mathcal{R}} E_{\mathcal{S}}} - e^{-\tilde{\beta}_{\mathcal{E}} E_{\mathcal{S}}} \right) + O(\lambda^3),
\end{aligned}$$

where

$$\kappa = Z^{-1}_{\beta_{\mathcal{R}}, \mathcal{S}} Z^{-1}_{\tilde{\beta}_{\mathcal{E}}, \mathcal{S}} \frac{\lambda_{\mathcal{R}}^2 \frac{\pi}{2} \sqrt{E_{\mathcal{S}}} \| f(\sqrt{E_{\mathcal{S}}}) \|^2 \lambda_{\mathcal{E}}^2 \tau \mathrm{sinc}^2(\tau(E_{\mathcal{S}} - E_{\mathcal{E}})/2)}{\lambda_{\mathcal{R}}^2 \pi \sqrt{E_{\mathcal{S}}} \| f(\sqrt{E_{\mathcal{S}}}) \|^2 + \lambda_{\mathcal{E}}^2 \tau \mathrm{sinc}^2(\tau(E_{\mathcal{S}} - E_{\mathcal{E}})/2)}$$

Remarks:

- $\kappa > 0$ and $\kappa = 0 \Leftrightarrow \lambda_{\mathcal{R}} \lambda_{\mathcal{E}} = 0$
- $dE_+^{\mathcal{C}} > 0$ if and only $T_{\mathcal{R}} = \beta_{\mathcal{R}}^{-1} > \tilde{T}_{\mathcal{E}} = \tilde{\beta}_{\mathcal{E}}^{-1}$ (leading order).
- $dS^+ \geq 0$ and $dS^+ = 0$ if and only if $T_{\mathcal{R}} = \tilde{T}_{\mathcal{E}}$ (leading order).
- dE_+^{tot} has no sign.

Elements of Proof

Reduction to a Product of Operators

$\varrho_0 = \langle\Psi_0| \cdot \Psi_0\rangle$ where $\Psi_0 = \Psi_{\mathcal{SR}} \otimes \Psi_{\mathcal{C}}$ with

$\Psi_{\mathcal{SR}} = \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{R}} \in \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}} \equiv \mathcal{H}_{\mathcal{SR}}$ and

$\Psi_{\mathcal{C}} = \Psi_{\mathcal{E}_1} \otimes \Psi_{\mathcal{E}_2} \otimes \cdots \in \mathcal{H}_{\mathcal{C}}$

$P = \mathbb{I}_{\mathcal{H}_{\mathcal{SR}}} \otimes |\Psi_{\mathcal{C}}\rangle\langle\Psi_{\mathcal{C}}|$ is the projector on $\mathcal{H}_{\mathcal{SR}} \otimes \mathbb{C}\Psi_{\mathcal{C}} \simeq \mathcal{H}_{\mathcal{SR}}$

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$C-$ Liouvillean Given $L_{\mathcal{S}} + L_{\mathcal{R}}$, $L_{\mathcal{E}}$ and $V_m \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}} \otimes \mathfrak{M}_{\mathcal{E}_m}$,

$$\exists \quad K_m \text{ s.t. } \begin{cases} e^{i\tilde{L}_m} A e^{-i\tilde{L}_m} = e^{iK_m} A e^{-iK_m} & \forall A \in \mathfrak{M}_{\mathcal{SR}} \otimes \mathfrak{M}_{\mathcal{C}} \\ K_m \Psi_{\mathcal{SR}} \otimes \Psi_{\mathcal{C}} = 0. \end{cases}$$

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$$K_m = \tau(L_{\text{free}} + V_m - V'_m), \quad V'_m = J_m \Delta_m^{\frac{1}{2}} V_m \Delta_m^{-\frac{1}{2}} J_m$$
$$:= \tau(L_{\text{free}} + \tilde{V}_m)$$

Tomita-Takesaki '57

Reduction to a Product of Matrices

Evolution of ϱ_0

$$\begin{aligned}
\varrho_0(\alpha^{m\tau}(A_{SR})) &= \langle\Psi_0|e^{i\tilde{L}_1}\cdots e^{i\tilde{L}_m}A_{SR}e^{-i\tilde{L}_m}\cdots e^{-i\tilde{L}_1}\Psi_0\rangle \\
&= \langle\Psi_0|e^{iK_1}\cdots e^{iK_m}A_{SR}e^{-iK_m}\cdots e^{-iK_1}\Psi_0\rangle \\
&= \langle\Psi_0|Pe^{iK_1}\cdots e^{iK_m}A_{SR}P\Psi_0\rangle \\
&= \langle\Psi_0|(Pe^{iK_1}P)(Pe^{iK_2}P)\cdots(Pe^{iK_m}P)A_{SR}\Psi_0\rangle \\
&\equiv \langle\Psi_{SR}|M_1M_2\cdots M_mA_{SR}\Psi_{SR}\rangle \\
&= \langle\Psi_{SR}|M^mA_{SR}\Psi_{SR}\rangle
\end{aligned}$$

where $M_j \simeq Pe^{iK_j}P$ on $\mathcal{H}_{SR}$ are all identical.

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Reduced Dynamical Operators

$M \in \mathcal{B}(\mathcal{H}_{\mathcal{SR}})$ s.t.

$$
\begin{cases}
M\Psi_{\mathcal{SR}} = \Psi_{\mathcal{SR}} \\
\|M^n\varphi\| \le C(\varphi), \quad \forall n \in \mathbb{N}, \quad \forall\varphi \text{ in a dense set}
\end{cases}
$$

Note: Ψ_{SR} cyclic and evolution is unitary.

Spectral Properties of RDO's

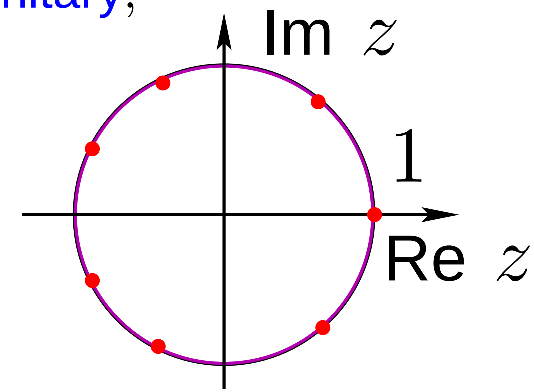
RDO

$$M = M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}$$

Uncoupled case

$$M_{(0,0)} = e^{i\tau(L_{\mathcal{S}}+L_{\mathcal{R}})}$$ unitary,

- eigenvalues of $M_{(0,0)}$: $\{e^{i\tau(e_k-e_l)}\}_{k,l}$
- 1 is $\dim \mathfrak{h}_{\mathcal{S}}$-fold degenerate
- ess spec $M_{(0,0)} = \mathbb{S}^1$



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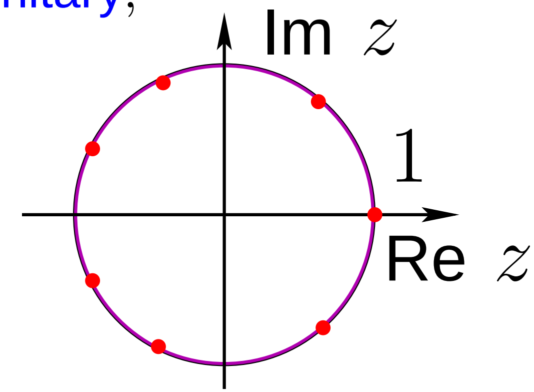

$(\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}}) \neq (0,0)$ $\Rightarrow$ Perturbation of embedded eigenvalues

$L_{\mathcal{R}} = d\Gamma(h)$ with h mult. by s on $L^2(\mathbb{R}, \mathcal{G})$ is suitable for translation analyticity Avron-Herbst 77

Translation Analyticity

Translation Group

$$\mathbb{R} \ni \theta \mapsto T(\theta) = \Gamma(e^{-\theta\partial_s}) \text{ on } \Gamma_-(L^2(\mathbb{R}, \mathcal{G}))$$

$$\text{s.t.} \quad (e^{-\theta\partial_s} f)(s) = f(s - \theta), \quad \forall f \in L^2(\mathbb{R}, \mathcal{G})$$

Assumption (A)

$\mathbb{R} \ni \theta \mapsto \tilde{V}_{\mathcal{SR}}(\theta) := T(\theta)^{-1} \tilde{V}_{\mathcal{SR}} T(\theta)$ admits an analytic extension to $\kappa_{\theta_0} = \{z \in \mathbb{C} \mid 0 < \text{Im } z < \theta_0\}$

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Recall

$$M = P\exp(iK)P\,, \quad \text{where}$$

$$K = \tau(L_0 + \lambda_{\mathcal{R}}\tilde{V}_{\mathcal{SR}} + \lambda_{\mathcal{E}}\tilde{V}_{\mathcal{SE}})\;, \quad L_0 = L_{\mathcal{S}} + L_{\mathcal{R}} + L_{\mathcal{E}}$$

Theorem The following op's are analytic $\forall\theta \in \kappa_{\theta_0}$

$$\begin{aligned} K(\theta) &= \tau(L_0 + \theta N + \lambda_{\mathcal{R}}\tilde{V}_{\mathcal{SR}}(\theta) + \lambda_{\mathcal{E}}\tilde{V}_{\mathcal{SE}}) \ \text{ on } \ D(L_0) \cap D(N), \\ M(\theta) &= P\exp(iK(\theta))P \in \mathcal{B}(\mathcal{H}_{\mathcal{SR}}) \end{aligned}$$

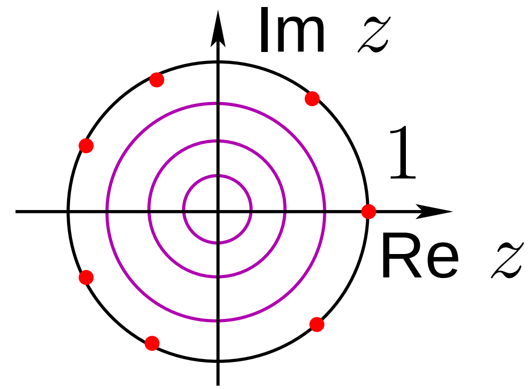
Translation Analyticity

Consequences

Discrete e.v. of $M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta)$ are θ-independent

Spectrum of $M_{(0,0)}(\theta) = \exp(i\tau(L_{\mathcal{S}} + L_{\mathcal{R}} + \theta N))$

$$\begin{cases} \text{eigenvalues of } M_{(0,0)}(\theta): \quad \{e^{i\tau(e_k - e_l)}\}_{k,l} \\ 1 \text{ is } \dim \mathfrak{h}_{\mathcal{S}}\text{-fold degenerate} \\ \text{ess spec } M_{(0,0)}(\theta) = \cup_{n=1}^{\infty}\{|z| = e^{-n\tau \mathrm{Im}\theta}\} \end{cases}$$



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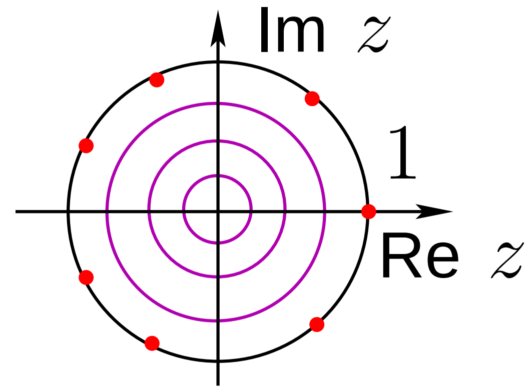
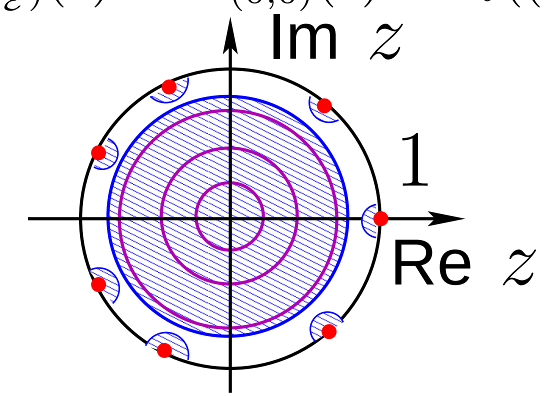


Perturbative approach

$$M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta) = M_{(0,0)}(\theta) + O_{\theta}((\lambda_{\mathcal{R}},\lambda_{\mathcal{E}}))$$



Lemma

$\|(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})\| < \lambda_0(\theta) \ \Rightarrow \sigma(M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta))$:

Asymptotic State

Analytic observables

$$A_{\mathcal{SR}} \text{ s.t. } A_{\mathcal{SR}}(\theta) = T(\theta)^{-1} A_{\mathcal{SR}} T(\theta) \text{ analytic in } \kappa_{\theta_0}$$

Note: For $A_{\mathcal{SR}}$ analytic,

$$\begin{aligned} \varrho_0(\alpha^{m\tau}(A_{SR})) &= \langle \Psi_{SR} | M^m A_{SR} \Psi_{SR} \rangle \\ &= \langle \Psi_{SR} | M(\theta)^m A_{SR}(\theta) \Psi_{SR} \rangle \end{aligned}$$

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Assumption (FGR)

$$\exists \theta_1 \in \kappa_{\theta_0}, \lambda_0(\theta_1) > 0 \text{ s.t. } \|(\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})\| < \lambda_0(\theta_1) \text{ implies}$$

$$\sigma(M_{(\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})}(\theta_1)) \cap \mathbb{S} = \{1\} \text{ and } 1 \text{ is simple}$$

Consequences

$$\lim_{n \to \infty} M_{(\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})}(\theta_1)^n = P_{1, M_{(\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})}(\theta_1)} = |\Psi_{\mathcal{SR}}\rangle \langle \psi^*_{(\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})}(\theta_1)|$$