V. ELECTRON GAS, RANDOM PHASE APPROXIMATION

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One of the most fundamental models in condensed matter theory is the homogeneous electron gas, a system of electrons interacting with a $1/r$-potential with each other to which a uniform positive background is added for charge neutrality. The Hamiltonian writes

$$H = H_0 + V \tag{1}$$

$$H_0 = \sum_{\mathbf{k}} \left(\frac{k^2}{2m} - \mu \right) a_k^\dagger a_k \tag{2}$$

$$V = \frac{1}{2\Omega} \sum_{\mathbf{q}\neq 0} v_q \sum_{\mathbf{k},\mathbf{k}'} a^\dagger_{\mathbf{k}+\mathbf{q}} a^\dagger_{\mathbf{k}'-\mathbf{q}} a_{\mathbf{k}'} a_{\mathbf{k}} \tag{3}$$

$$= \frac{1}{2\Omega} \sum_{\mathbf{q}\neq 0} v_q \left[\rho_{\mathbf{q}} \rho_{-\mathbf{q}} - \rho_0 \right] \tag{4}$$

where $a^\dagger_{\mathbf{k}}$ ($a_{\mathbf{k}}$) are creation (annihilation) operators with the following anticommutation relations

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}]_+ \equiv a_{\mathbf{k}} a_{\mathbf{k}'} + a_{\mathbf{k}'} a_{\mathbf{k}} = 0, \quad \left[a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}\right]_+ = 0, \quad \left[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}\right]_+ = \delta_{\mathbf{k},\mathbf{k}'} \tag{5}$$

and the density operator is introduced via

$$\rho_{\mathbf{q}} = \sum_{\mathbf{k}} a^\dagger_{\mathbf{k}+\mathbf{q}} a_{\mathbf{k}} \tag{6}$$

(Note that $\rho_0 = N$ where N is the total number operator.) The Fourier transform of the $1/r$ potential in $d = 2$ or $d = 3$ dimensions writes

$$v_q = \frac{2(d-1)\pi e^2}{q^{d-1}} \tag{7}$$

where e is the electronic charge and Ω the volume of the system. Note that in the interaction part of the Hamiltonian, Eq. (3), excludes the term with $q = 0$ in the summation which accountes for the uniform positive charged background. Further, assuming periodic boundary conditions, the Coulomb potential can only be written down in Fourier space due to its long-range behavior. (In real space one must assume a screened Coulomb-potential $e^{-\alpha r}/r$ and take the limit $\alpha \to 0$ at the very end of the calculation to exclude non-neglegible surface-effects.)

At zero temperature the electron gas is characterized by one dimensionless parameter r_s

$$r_s = a/a_B \tag{8}$$

where a is roughly the mean interparticle distance, or

$$\Omega = N \frac{2(d-1)\pi a^d}{d} \tag{9}$$

and $a_B = \hbar^2/(me^2)$ is the Bohr radius

$$\frac{\hbar^2}{m a_B^2} = \frac{e^2}{a_B} \tag{10}$$

Energies are convieniently given in Rydbergs, $1Ry = \hbar^2/(2ma_B^2) = me^4/2\hbar^2$. Let us estimate the kinetic and potential energy. The kinetic energy, T, is given by the Fermi energy

$$\frac{T}{N} \sim \frac{k_F^2}{2m} \sim \frac{1}{2ma^2} = \frac{1}{r_s^2} Ry \tag{11}$$

where $k_F \sim a^{-1}$ is the Fermi wavevector. The potential energy is roughly

$$\frac{V}{N} \sim \frac{e^2}{a} = \frac{2}{r_s} Ry \tag{12}$$

We see, that in the high density limit $n \to \infty$, or $r_s \to 0$, the kinetic energy dominates the potential energy. In this limit one might expect that the electron-electron interaction V can be treated as a perturbation of the non-interacting ground state.

A. Equation of motions approach

Let us consider that the system is in the ground state of the non-interacting Hamiltonian,

$$|F\rangle = \prod_{\mathbf{k} \leq k_F} a_{\mathbf{k}}^\dagger |0\rangle \tag{13}$$

where the Fermi-wavevector k_F is related to the density

$$n = \frac{1}{\Omega} \sum_{\mathbf{k}} \langle F | a_{\mathbf{k}}^\dagger a_{\mathbf{k}} | F \rangle = \frac{1}{\Omega} \sum_{|\mathbf{k}| \leq k_F} 1 \tag{14}$$

and we consider a spin-polarized system for simplicity.

1. *Single particle excitation*

Let us create an additional particle, and add it to the system. In the Heisenberg picture, the time evolution of the creation operator $a_{\mathbf{k}}^\dagger$ with $|\mathbf{k}| > k_F$ is given by

$$-i\frac{d}{dt} a_{\mathbf{k}}^\dagger = \left[H, a_{\mathbf{k}}^\dagger \right]_- = \omega_k a_{\mathbf{k}}^\dagger + \frac{1}{\Omega} \sum_{\mathbf{q} \neq 0} \sum_{\mathbf{p}} v_q a_{\mathbf{k}+\mathbf{q}}^\dagger a_{\mathbf{p}-\mathbf{q}}^\dagger a_{\mathbf{p}} \tag{15}$$

where we have defined $\omega_k^{(0)} = k^2/2m - \mu$ and used the anti-commutation rules. Now we separate on the rhs. the terms which involves $a_{\mathbf{k}}^\dagger$, namely $\mathbf{p} - \mathbf{q} = \mathbf{k}$. We get

$$-i\frac{d}{dt} a_{\mathbf{k}}^\dagger = \left[H, a_{\mathbf{k}}^\dagger \right]_- = \left(\omega_k^{(0)} - \frac{1}{\Omega} \sum_{\mathbf{q} \neq 0} v_q a_{\mathbf{k}+\mathbf{q}}^\dagger a_{\mathbf{k}+\mathbf{q}} \right) a_{\mathbf{k}}^\dagger + \frac{1}{\Omega} \sum_{\mathbf{q} \neq 0} \sum_{\mathbf{p} \neq \mathbf{k}+\mathbf{q}} v_q a_{\mathbf{k}+\mathbf{q}}^\dagger a_{\mathbf{p}-\mathbf{q}}^\dagger a_{\mathbf{p}} \tag{16}$$

Let us assume the simple time evolution

$$a_{\mathbf{k}}^\dagger(t) = a_{\mathbf{k}}^\dagger(0) e^{i\omega_k^{(1)} t} \tag{17}$$

we obtain

$$\omega_k^{(1)} a_{\mathbf{k}}^\dagger = \left(\omega_k^{(0)} - \frac{1}{\Omega} \sum_{\mathbf{q} \neq 0} v_q a_{\mathbf{k}+\mathbf{q}}^\dagger a_{\mathbf{k}+\mathbf{q}} \right) a_{\mathbf{k}}^\dagger + \frac{1}{\Omega} \sum_{\mathbf{q} \neq 0} \sum_{\mathbf{p} \neq \mathbf{k}+\mathbf{q}} v_q a_{\mathbf{k}+\mathbf{q}}^\dagger a_{\mathbf{p}-\mathbf{q}}^\dagger a_{\mathbf{p}} e^{i(\omega_{\mathbf{k}+\mathbf{q}}^{(1)} + \omega_{\mathbf{p}-\mathbf{q}}^{(1)} - \omega_{\mathbf{p}}^{(1)} - \omega_{\mathbf{k}}^{(1)})t} \tag{18}$$

Since the last term on the rhs is always oscillating, we expect that it does not controbute for times much longer compared to typical energies $\omega_k^{(1)}$ and vanishes on average. Approximations in this spririt are frequently called

Random Phase Approximation (RPA). However, in the context of the electron gas, this is not called RPA! Replacing $a_k^\dagger a_k$ by its expectation value n_k we recover the Hartree-Fock single particle excitation spectrum

$$\omega_k^{(1)} = \omega_k^{(0)} - \frac{1}{\Omega}\sum_{\mathbf{q}\neq 0} v_q n_{\mathbf{k}+\mathbf{q}} \tag{19}$$

since the Hartree term vanishes due to the positive charged background.

Note that the Random-Phase-approximation for the single particle excitations averaged away the last term on the rhs of Eq. (18) which contains the diverging potential in the limit $q \to 0$. This is a-priori not justified.

2. Collective excitations

The a-priori candidats for collective excitations are density fluctuations, described by the density operator $\rho_{\mathbf{k}} = \sum_{\mathbf{p}} a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}}$. For the equation of motion we need the following commutators

$$\left[H_0, a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}}\right]_- = \left(\varepsilon_{\mathbf{p}+\mathbf{k}} - \varepsilon_{\mathbf{p}}\right) a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}} \tag{20}$$

$$\left[\rho_{\mathbf{q}}, \rho_{\mathbf{k}}\right]_- = 0 \tag{21}$$

$$\left[\rho_{\mathbf{q}}, a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}}\right]_- = a^\dagger_{\mathbf{p}+\mathbf{k}+\mathbf{q}} a_{\mathbf{p}} - a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}-\mathbf{q}} \tag{22}$$

We get

$$(-i)^2 \frac{d^2}{dt^2}\rho_{\mathbf{k}} = \sum_{\mathbf{p}} \left(\varepsilon_{\mathbf{p}+\mathbf{k}} - \varepsilon_{\mathbf{p}}\right) \left[H, a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}}\right]_- \tag{23}$$

$$= \sum_{\mathbf{p}} \frac{2\mathbf{p}\cdot\mathbf{k} + k^2}{2m} \left[H, a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}}\right]_- \tag{24}$$

$$= \sum_{\mathbf{p}} \left(\frac{2\mathbf{p}\cdot\mathbf{k} + k^2}{2m}\right)^2 a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}} + \frac{1}{2\Omega}\sum_{\mathbf{p},\mathbf{q}\neq 0} \frac{2\mathbf{p}\cdot\mathbf{k} + k^2}{2m} v_q \left[\rho_{-\mathbf{q}}\rho_{\mathbf{q}}, a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}}\right]_- \tag{25}$$

$$= \sum_{\mathbf{p}} \left(\frac{2\mathbf{p}\cdot\mathbf{k} + k^2}{2m}\right)^2 a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}} + \frac{1}{2\Omega}\sum_{\mathbf{p},\mathbf{q}\neq 0} \frac{2\mathbf{p}\cdot\mathbf{k} + k^2}{2m} v_q \left\{\rho_{-\mathbf{q}}\left(a^\dagger_{\mathbf{p}+\mathbf{k}+\mathbf{q}} a_{\mathbf{p}} - a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}-\mathbf{q}}\right) + \left(a^\dagger_{\mathbf{p}+\mathbf{k}-\mathbf{q}} a_{\mathbf{p}} - a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}+\mathbf{q}}\right)\rho_{\mathbf{q}}\right\} \tag{26}$$

$$= \sum_{\mathbf{p}} \left(\frac{2\mathbf{p}\cdot\mathbf{k} + k^2}{2m}\right)^2 a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}} + \frac{1}{\Omega}\sum_{\mathbf{p},\mathbf{q}\neq 0} \frac{2\mathbf{p}\cdot\mathbf{k} + k^2}{2m} v_q \rho_{-\mathbf{q}} \left(a^\dagger_{\mathbf{p}+\mathbf{k}+\mathbf{q}} a_{\mathbf{p}} - a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}-\mathbf{q}}\right) \tag{27}$$

where we have used $[AB, C]_- = A[B, C]_- + [A, C]_- B$. Relabelling $\mathbf{p} \to \mathbf{p} - \mathbf{q}$ in the last term on the rhs we get

$$(-i)^2 \frac{d^2}{dt^2}\rho_{\mathbf{k}} = \sum_{\mathbf{p}} \left(\frac{2\mathbf{p}\cdot\mathbf{k} + k^2}{2m}\right)^2 a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}} - \frac{1}{\Omega}\sum_{\mathbf{q}\neq 0} \frac{2\mathbf{q}\cdot\mathbf{k}}{2m} v_q \rho_{-\mathbf{q}}\rho_{\mathbf{k}+\mathbf{q}} \tag{28}$$

In order to prepare the RPA-approximation, we separate the terms linear in $\rho_{\mathbf{k}}$ on the rhs, that means the term $\mathbf{q} = -\mathbf{k}$ in the summation. Noting that the density $n = \rho_0/\Omega$, we have

$$(-i)^2 \frac{d^2}{dt^2}\rho_{\mathbf{k}} = \frac{nk^2 v_k}{m}\rho_{\mathbf{k}} + \sum_{\mathbf{p}} \left(\frac{2\mathbf{p}\cdot\mathbf{k} + k^2}{2m}\right)^2 a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}} - \frac{1}{\Omega}\sum_{\mathbf{q}\neq\{0,-\mathbf{k}\}} \frac{\mathbf{q}\cdot\mathbf{k}}{m} v_q \rho_{-\mathbf{q}}\rho_{\mathbf{k}+\mathbf{q}} \tag{29}$$

Now in the limit $\mathbf{k} \to 0$ (long wavelength fluctuations), the first term on the rhs can be written in terms of the plasma frequency, ω_p,

$$\omega_p^2 = \lim_{k\to 0} \frac{nk^2 v_k}{m} \tag{30}$$

In three dimensions, since $v_k \to k^{-2}$, the plasma-frequcency is independant of the wavevector to first approximation, $\omega_p = (4\pi ne^2/m)^{1/2}$ Noting that $p \approx k_F$ in second term on the rhs of Eq. (29), and averaging over the angles it can be approximated as $\approx k_F^2 k^2 \rho_{\mathbf{k}}/3m^2$. Writing the time evolution of the density operator as

$$\rho_{\mathbf{k}}(t) = \rho_{\mathbf{k}}(0) e^{i\omega_p(k)t} \tag{31}$$

we obtain a self-consistent solution for $k \to 0$ with

$$\omega_p^2(k) = \omega_p^2 + k_F^2 k^2/3m^2 \tag{32}$$

since the last term on the rhs of Eq. (29) is oscillating in time and can be neglected in the long-time evolution in the spirit of the RPA. We expect that the plasma-oscillations are good eigenstates for

$$k \leq k_c \approx m\omega_p/k_F \tag{33}$$

In three dimensions we have $k_c a \sim r_s^{1/2}$.

B. Ground state wavefunction

Our results concerning the single particle and collective excitations propose the following ground state many-body wavefunction of the interacting electron gas in the high density limit

$$\Psi_{RPA}(\mathbf{R}) = \mathcal{N}^{-1/2} \exp\left[-(2\Omega)^{-1} \sum_{\mathbf{q}} \alpha_q \rho_{\mathbf{q}} \rho_{-\mathbf{q}}\right] \det_{\mathbf{k},i} e^{i\mathbf{k}\cdot\mathbf{r}_i} \tag{34}$$

where $\mathcal{N}$ accounts for the correct normalization. Since we have seen that $a_k^\dagger$ creates a single particle excitation ($k > k_F$), or a_k creates a hole for $k \leq k_F$, we expect that the determinant is build out of plane waves with wavevectors $k \leq k_F$. Since the collective excitations where described by a single harmonic oscillator behavior, $\rho_{\mathbf{q}}$ plays the role similar to that of the coordinate (or momentum) operator of the usual harmonic oscillator, and we expect a harmonic oscillator ground state wavefunction for $\rho_{\mathbf{q}}$ for each oscillator $\mathbf{q}$. We just need to determine α_q. Note, since $\rho_{\mathbf{q}} = \sum_i e^{-\mathbf{q}\cdot\mathbf{r}_i}$ in first quantization, the wavefunction is of the Slater-Jastrow type, since it can be written as

$$\Psi_{RPA}(\mathbf{R}) = \mathcal{N}^{-1/2} \exp\left[-\sum_{i,j} \alpha(|\mathbf{r}_i - \mathbf{r}_j|)\right] \det_{\mathbf{k},i} e^{i\mathbf{k}\cdot\mathbf{r}_i} \tag{35}$$

where $\alpha(r)$ is just the Fourier transform of $\alpha_{\mathbf{q}}$

In order to determine α_q we apply the Hamiltonian in first quantization on our variational wavefunction, Eq. (34),

$$E_L(\mathbf{R}) \equiv \frac{H\Psi_{RPA}(\mathbf{R})}{\Psi_{RPA}(\mathbf{R})} \tag{36}$$

$$= \sum_{k\leq k_F} \frac{k^2}{2m} + \frac{1}{2\Omega} \sum_{\mathbf{q}\neq 0} \left[v_q(\rho_q\rho_{-q} - N) + \frac{2q^2\alpha_q}{2m}(\rho_q\rho_{-q} - N) + \frac{2}{\Omega} \sum_{q'} \mathbf{q}\cdot\mathbf{q}' \frac{\alpha_q\alpha_{q'}}{2m} \rho_{\mathbf{q}+\mathbf{q}'}\rho_{-\mathbf{q}}\rho_{-\mathbf{q}'} \right] \tag{37}$$

We have neglected a cross-term between the Jastrow and the Slater-determinant. Now within the RPA, only the terms in the last summation of the rhs with $q = 0$, $q' = 0$ or $\mathbf{q} + \mathbf{q}' = 0$ must be kept, an we get

$$E_L(\mathbf{R}) = \sum_{k\leq k_F} \frac{k^2}{2m} + \frac{1}{2\Omega} \sum_{\mathbf{q}\neq 0} \left[v_q(\rho_q\rho_{-q} - N) + \frac{q^2\alpha_q}{m}(\rho_q\rho_{-q} - N) - \sum_q \frac{n(q\alpha_q)^2}{m} \rho_{\mathbf{q}}\rho_{-\mathbf{q}} \right] \tag{38}$$

In the limit $q \to 0$ we have to cancel the Coulomb singularity and we get

$$\alpha_q^2 = \frac{mv_q}{nq^2} = \frac{m^2 nq^2 v_q}{mn^2 q^4} \tag{39}$$

This determines the exact value of α_q for $q \to 0$. It is the basic structure of many-body wavefunctions of Coulomb systems in the Quantum-Monte Carlo approach.

Bohm and Pines have used the separation of plasmon and single particle excitation introducing additional collective excitations to descrie the plasmons. We have already choosen the parameters α_q of the wavefunction to remove the Coulomb singularity. The resulting effective potential is then screened by charge fluctuations and the effective potential can be treated perturbatively.

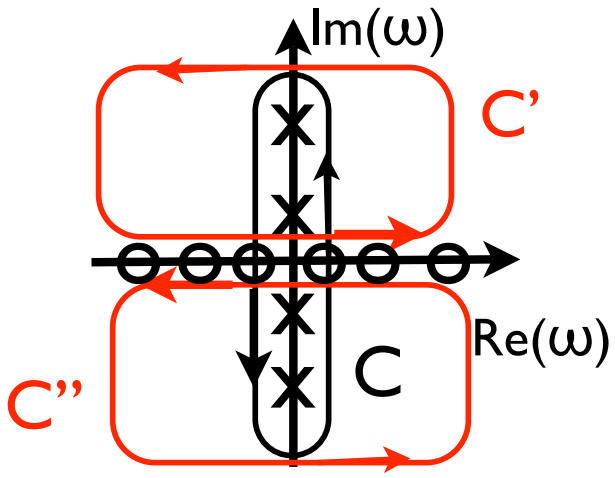


FIG. 1: A deformation of the contour integration C to C' and C''. Crosses are the Matsubara frequencies, the poles of the Fermi-function, whereas circles indicate possible poles (or a branch cut) of the original function. Note that C' is just above and C'' just below the real axis.

C. Green's function approach - RPA diagrams

In the equation of motion approach, we have already noticed that the divergence of the Coulomb potential can be dangereous. Within the Green's function approach, we can analyse the potentially diverging diagrams. We can see that the bubble-diagrams, $B(q)$, can be dangereous, as thay occur in the form $[-v_q B_0(q)]^n$, with an arbitrary high power n. In the case, where the simple bubble $B_0(q)$ is finite at $q \to 0$, we are forced to include a whole series in order to avoid divergence. The resummation of this series is the RPA approximation in the diagrammatic approach.

We need a basic method to evaluate summation over Matsubara-frequencies $\omega_n = \pi(2n+1)T$ for $n = 0, \pm 1, \ldots$

$$\sum_n g(\omega_n) = -\frac{\beta}{2\pi i} \lim_{\tau \to 0} \int_C e^{\omega\tau} f_F(\omega) g(\omega) d\omega \tag{40}$$

$$f_F(\omega) = \frac{1}{e^{\beta\omega}+1} \tag{41}$$

where the integral is in the complex planes with a contour C going around each pole of the Fermi-function $f_F(\omega)$ at $\omega = \omega_n$ in the positive sense. If we know the analytic properties, we can deform the contour to evaluate the Matsubara summation. (The $e^{i\omega 0+}$-factor is often there, but not explicitly written down.)

Simple branch cut on the real axis. Suppose that $g(\omega)$ is analytic in the upper/ lower complex plane, but with simple poles on the real axis. Further, $g(\omega)$ vanishes for $|\omega| \to \infty$. We can deform now the integration contour into two half-circles, one in the upper complex plane, one in the lower complex plane. The Fermi function assures that the integration at infinity with positive real part of ω is neglegible, whereas the $e^{\omega 0^+}$ is needed to neglect the contributions at infinity for negative real part of ω. Only both integral close to the real axis contribute

$$T\sum_n g(i\omega_n) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega d\omega e^{\omega 0^+} f_F(\omega) \left[g(\omega + i\eta) - g(\omega - i\eta)\right] = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} d\omega e^{\omega 0^+} f_F(\omega) A(\omega) \tag{42}$$

where we introduced the spectral function

$$A(\omega) = -\mathrm{Im}\, g(\omega + i\eta) \tag{43}$$

The spectral function is a fundamental quantity, since it occurs in many expressions. In the case of free fermions, we have $G_0(k,\omega_n)^{-1} = i\omega_n + \mu - k^2/2m$. The analytic continuation for ω away from the real axis is simple given by replacing $i\omega_n$ by ω. The spectral function is then

$$A_0(k,\omega) = 2\pi\delta(\omega - k^2/2m + \mu) \tag{44}$$

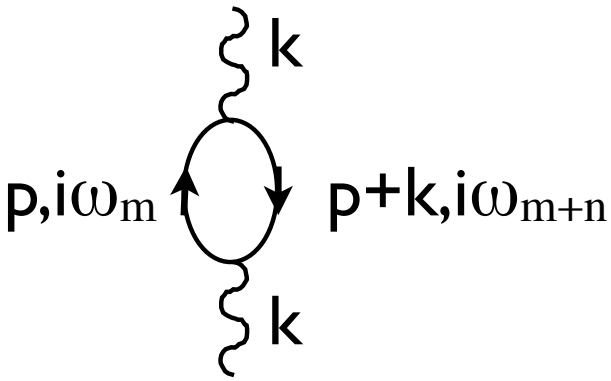


FIG. 2: The basic bubble diagram.

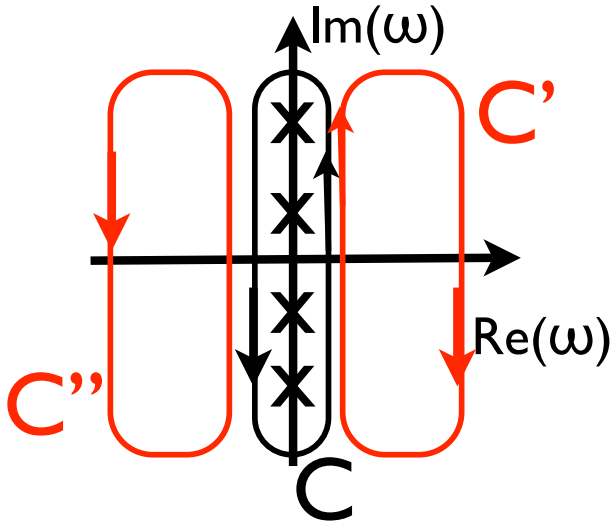


FIG. 3: A different deformation of the contour integration which we use to calculate the bubble diagram.

using the formal identity $1/(x+i\eta) = P/x - i\pi\delta(x)$. Putting the spectral function of the ideal Fermi gas bacl into the expression (42), we recover the density of the ideal Fermi gas. Note, that we could have also used different contours to evaluate the Matsubara summation leading to the same final result.

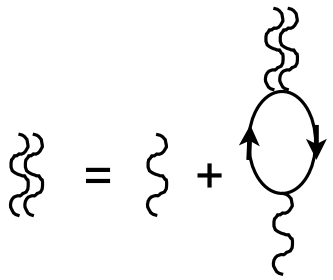
Bubble diagram. Let us calculate the basic bubble diagram

$$B_0(\mathbf{k}, i\omega_n) = T\int \frac{d\mathbf{p}}{(2\pi)^3}\sum_m G_0(\mathbf{p}+\mathbf{k}, i\omega_{m+n})G_0(\mathbf{p}, i\omega_m) \tag{45}$$

$$= \int \frac{d\mathbf{p}}{(2\pi)^3}\int_C \frac{d\omega}{2\pi i} f_F(\omega)\frac{1}{\omega + 2\pi i n T + \mu - \varepsilon_{\mathbf{k}+\mathbf{p}}}\frac{1}{\omega+\mu-\varepsilon_p} \tag{46}$$

We can deform the contour and close it to the right and to the left, separately. The contributions from the half-circles with $|\omega| \to \infty$ vanishes, and we have only to take into account the two poles at $\omega = \varepsilon_{\mathbf{k}+\mathbf{q}} - \mu - 2\pi n T$ and at $\omega = \varepsilon_p - \mu$. We get

$$B_0(\mathbf{k}, i\omega_n = 2\pi i n T) = \int \frac{d\mathbf{p}}{(2\pi)^3}\frac{f_F(\varepsilon_p - \mu) - f_F(\varepsilon_{\mathbf{k}+\mathbf{p}} - \mu)}{i\omega_n + \varepsilon_p - \varepsilon_{\mathbf{p}+\mathbf{k}}} \tag{47}$$



$$v_{eff}(q)=v(q)+v(q)B_0(q)\ v_{eff}(q)$$

FIG. 4: The RPA approximation for the effective interaction.

where we have used that $f(\omega + 2\pi nT) = f(\omega)$. Note that the bubbles is now bosonic, concerning at least the Matsubara-index! Just note that we have treated the spin-polarized case, for unpolarized electrons the bubble result has to be multiplied by a factor of 2.

Screened Coulomb interaction - RPA summation. We can now analyse the bubble diagram in the limit $k \to 0$ in order to estimate the order of the divergence higher order perturbation terms invoving the structure $[-v_q B_0(q)]^n$. For zero Matsubara frequency,

$$B_0(k \to 0, 0) = \lim_{k\to 0} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{f_F(\varepsilon_p - \mu) - f_F(\varepsilon_{\mathbf{k}+\mathbf{p}} - \mu)}{\varepsilon_p - \varepsilon_{\mathbf{p}+\mathbf{k}}} = -\beta \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{df_F(\varepsilon_{\mathbf{k}} - \mu)}{d(\beta\mu)} = -\frac{dn}{d\mu} \tag{48}$$

and we get a finite negative number, proportional to the compressibility of the free Fermi gas. At zero temperature it reduces to the density of states at the Fermi level, k_F. For any finite Mastubara frequency, $B_0(k, i\omega_n \neq 0) \sim k^2$. Therefore we get singularities of order $[-v(q)B_0(q)]^n \sim 1/q^{2n}$ from a chain of n bubbles. However, it is bossible to include the bubbles sum up to all order, introducing the effective potential

$$v_{eff}(q) = v_q - v_q B_0(q, 0) + [v_q B_0(q, 0)]^2 - \cdots = \frac{v_q}{1 - v_q B_0(q, 0)} \tag{49}$$

For $q \to 0$, we obtain a screened Coulomb potential in three dimensions,

$$v_{eff}(q) \to \frac{4\pi e^2}{q^2 + k_s^2}, \quad k_s^2 = 4\pi e^2 (dn/d\mu) \tag{50}$$

Two-Particle Green's function, Dielectric function. The bubble-diagram contains a particle-hole propagator which represents a vertex contribution to the two-particle Green's function. Schematically we have

$$G^{(2)}(1, 2; 1', 2') = G(1, 1')G(2, 2') - G(1, 2')G(2, 1') + G(1, A)G(2, B)\Gamma(A, B; A', B')G(A', 1')G(B', 2') \tag{51}$$

The RPA summation can be seen as the following approximation for the vertex Γ,

$$\Gamma = v + vB_0\Gamma \tag{52}$$

which leads to the screened interaction $\Gamma = v/(1 - vB_0) = 1/(v^{-1} - B_0)$. It is straightforward to see, that an external potential will be also screened. A particular form of the two-particle Green's function is the density-density correlation function, from which the dielectric function can be determined.

Fermi Liquid Theory. Let us look at the exact one-particle Green's function which can be written as

$$G^{-1}(\mathbf{k}, z) = z - \zeta_k - \Sigma(\mathbf{k}, z) \tag{53}$$

$$\zeta_k = \varepsilon_k - \mu \tag{54}$$

We have seen before, that the poles of the analytic continuation, in particular the spectral function, enters into the Fermi-function in a similar way as the exact single particle energies for the ideal Fermi gas. Let us continue approach the real axis from above for the self-energy

$$\Sigma(\mathbf{k}, \omega) \equiv \Sigma(\mathbf{k}, z = \omega + i\eta) \tag{55}$$

and separate real and imaginary parts

$$G^{-1}(\mathbf{k},\omega) \;=\; \omega - [\zeta_k + \mathrm{Re}\,\Sigma(\mathbf{k},\omega)] - i\,\mathrm{Im}\,\Sigma(\mathbf{k},\omega) \tag{56}$$

The poles of the Green's function, and thus the single particle excitation energies, are determined by $G^{-1}(\mathbf{k},\epsilon_k)=0$. We might expect that $\epsilon_k \approx \zeta_k \approx 0$. Neglecting the imaginary part of the self-energy for a moment, we expect that we can expand the real part of the self-energy around ζ_{k_F}, and k around k_F. We get

$$G(\mathbf{k},\omega) \;\approx\; \frac{Z}{\omega - \widetilde{\zeta}_k} \tag{57}$$

$$Z^{-1} \;=\; 1 - \left.\frac{\partial \mathrm{Re}\,\Sigma(k_F,\omega)}{\partial\omega}\right|_{\omega=0} \tag{58}$$

$$\widetilde{\zeta}_k \;=\; \frac{(k-k_F)k_F}{m^*} \tag{59}$$

$$\frac{1}{m^*} \;=\; \frac{Z}{k_F}\left.\frac{\partial[\zeta_k + \mathrm{Re}\,\Sigma(k_F,0)]}{\partial k}\right|_{k=k_F} \tag{60}$$

In contrast to the ideal gas where $\zeta_k \simeq (k-k_F)k_F/m$ close to k_F, the system behaves similar to an ideal gas, but with an effective mass m^*. Further, the weight of the excitation is reduced from one to $Z \leq 1$, but we still have a delta-peak in the spectral function writes

$$A(\mathbf{k},\omega) = 2\pi Z \delta(\omega - \widetilde{\zeta}_k) \tag{61}$$

However the general sum rule

$$\int \frac{d\omega}{2\pi} A(k,\omega) = 1 \tag{62}$$

is not anymore satisfied, due to a incoherent part in the spectral function which is connected to the imaginary part of the self energy.

In general, these expansions are only valid if the imaginary part of the self energy is neglegible. One might expect that this is the case for the electron gas for excitations close to the Fermi-surface, since the Pauli-factors reduce dramatically the avaiable phase space for incoherent scattering. Whenever, the imaginary part is sufficiently small, we can speak of a Fermi liquid, where many properties behave similar to the ideal Fermi gas.

Wigner Crystal. All the previous calculations relied upon the adiabatique deformation of the non-interacting Fermi liquid ground state due to the Coulomb interaction, reasonable for the high density gas, $r_s \to 0$. However, at low density, this assumtion breaks down. The interaction dominates and the electrons start to avoid each other forming a regular crystal (bcc in three dimensions). Roughly, the kinetic energy cost for localizing one electron inside a volume of order of the elementary cell of the lattice is neglegible compared to the potential energy gain. This is the classical limit.

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