V. FERMI LIQUID THEORY

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In general, the straighforward application of perturbation theory is rather limited. Resummations of subseries have to be performed in order to derive results for interacting matter. Very often, these resummations are physically motivated, and it is difficult to judge its reliability. Landau's approach for strongly interacting Fermi liquid provides a complementary approach where the fundamental degrees of freedom are expressed in terms of few phenomenological parameters. Here we present a few results of the heuristic theory, the microscopic justification is given later.

A. Ideal Fermi gas

Let us start by considering some essential elements characterizing the non-interacting Fermi gas. The ground state of spin 1/2 Fermions is given by occupying plane wave states up to the Fermi momentum k_F which is determined by the density

$$\rho = 2\int_0^{k_F} \frac{d^3\mathbf{k}}{(2\pi)^3} = \frac{k_F^3}{3\pi^2} \tag{1}$$

The ground state can be characterized by the occupation number, $n_0(\mathbf{k})$ of state $\mathbf{k}$,

$$n_0(\mathbf{k}) = 2\theta(k_F - |\mathbf{k}|) \tag{2}$$

and the total ground state energy is given by

$$E = \sum_{\mathbf{k}} \frac{\hbar^2 k^2}{2m} n_0(\mathbf{k}) \tag{3}$$

The excited states of the ideal gas are entirely characterized by the change of the occupation number $\delta n(\mathbf{k}) = n(\mathbf{k}) - n_0(\mathbf{k})$, e.g. the energy of the excitation is

$$\delta E = \sum_{\mathbf{k}} \frac{\hbar^2 k^2}{2m} \delta n(\mathbf{k}) \tag{4}$$

Changes of the paramters of the system, e.g. temperature, external fields, can be expressed in terms of $\delta n(\mathbf{k})$. In particular, low-energy properties, are entirely determined by the structure around the Fermi wavevector, k_F.

B. Fermi Liquid Theory

Suppose, we turn on the interacting between the particles, adiabatically. Landau's Fermi liquid theory aims on the description of the excited states with low energies. We consider the situation where, inside a certain region of energy, a one-to-one correspondence between eigenstates of the free Fermi gas and those of the interacting system exists (not neccessarily very close to the ground state). In that case, these states can be characterized by the change of the occupation number, as in the ideal gas, and lead to the occupation of quasi-particules and -holes, or, equivalently, the total energy of an excited state is a functional of the quasiparticle distribution function, $E\{n_{\mathbf{k}\sigma}\}$. The quasi-particle energy, $\varepsilon_{\mathbf{k}\sigma}\{n_{\mathbf{p}\sigma'}\}$ can then be defined as the variation of E with $n_{\mathbf{k}\sigma}$

$$\delta E = \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}\sigma} \delta n_{\mathbf{k}\sigma} \tag{5}$$

Note that the quasiparticle energy is itself a functional of the distribution function.

In a macroscopic state of thermal equilibrium, any variation of the thermodynamic equilibrium at finite temperture is given by

$$\delta E = T\delta S + \mu\delta N \tag{6}$$

where δS is the variation of the entropy, δN the variation of the particle number, and T the temperature, μ the chemical potential. Postulating a one-to-one correspondence of the eiegenstates with those of an ideal gas, the entropy must have the same form as for the free Fermi gas

$$S = -k_B \sum_{\mathbf{p}\sigma} \left[n_{\mathbf{p}\sigma} \ln n_{\mathbf{p}}\sigma + (1 - n_{\mathbf{p}\sigma}) \ln(1 - n_{\mathbf{p}\sigma}) \right] \tag{7}$$

where k_B is the Boltzmann's constant. Further the total number of quasiparticles is the same as the number of particles of the ideal gas

$$N = \sum_{\mathbf{p}\sigma} n_{\mathbf{p}\sigma} \tag{8}$$

Now, let us consider the variation, $\delta n_{\mathbf{p}\sigma}$ of the quasi-particle occupation number. The change in the entropy is then

$$\delta S = -k_B \sum_{\mathbf{p}\sigma} \delta n_{\mathbf{p}\sigma} \ln \frac{n_{\mathbf{p}\sigma}}{1 - n_{\mathbf{p}\sigma}} \tag{9}$$

and the variation of the total number of particles

$$\delta N = \sum_{\mathbf{p}\sigma} \delta n_{\mathbf{p}\sigma} \tag{10}$$

and the resulting change of the energy is given by

$$\delta E = T\delta S + \mu\delta N = \sum_{\mathbf{p}\sigma} \left[-k_B T \ln \frac{n_{\mathbf{p}\sigma}}{1 - n_{\mathbf{p}\sigma}} + \mu \right] \delta n_{\mathbf{p}\sigma} \tag{11}$$

and equating with the definition of the quasiparticle energy, Eq. (5), we get

$$\varepsilon_{\mathbf{p}\sigma} - \mu = k_B T \ln(n_{\mathbf{p}\sigma}^{-1} - 1) \tag{12}$$

which gives the usual Fermi-Dirac distribution for the occupation number

$$n_{\mathbf{p}\sigma} = \frac{1}{e^{\beta(\varepsilon_{\mathbf{p}\sigma} - \mu)} - 1} \tag{13}$$

Since the quasiparticle energy, $\varepsilon_{\mathbf{p}\sigma}$ is itself a functional of the distribution function, this is only a complicated implicit equation for $n_{\mathbf{p}\sigma}$.

Effective mass, density of states. At $T = 0$, we get

$$n^0_{\mathbf{p}\sigma} = \theta(\mu - \varepsilon^0_{\mathbf{p}\sigma}) \tag{14}$$

and μ equals $\varepsilon^0_{k_F}$ at the Fermi surface. For slight perturbations close to zero temperature, the occupation number can only vary in the neighborhood of k_F. Expanding to first order we get

$$\varepsilon^0_{\mathbf{p}\sigma} = \mu + v_F(p - p_F) \tag{15}$$

where

$$v_F = \left. \frac{\partial \varepsilon^0_{\mathbf{p}\sigma}}{\partial p} \right|_{\mathbf{p}=p_F} = \frac{p_F}{m^*} \tag{16}$$

defines the effective mass, m^*. The difference of the effective mass to the bare mass will modify the density of quasiparticle states compared to the ideal gas

$$N(0) = \frac{1}{V} \sum_{\mathbf{p}\sigma} \delta(\varepsilon^0_{\mathbf{p}\sigma} - \mu) = 2 \int \frac{d^3\mathbf{p}}{(2\pi)^3} \delta(\varepsilon^0_{\mathbf{p}\sigma} - \mu) = \frac{1}{\pi^2} \int p^2 \delta(\varepsilon^0_{\mathbf{p}\sigma} - \mu) dp \tag{17}$$

Since $d\varepsilon^0_{\mathbf{p}\sigma}/dp = p_F/m^*$ and $p = p_F$ at $\varepsilon^0_{\mathbf{p}\sigma} = \mu$, we get

$$N(0) = \frac{m^* p_F}{\pi^2} \tag{18}$$

Quasiparticle interaction. In general, quasiparticules will interact among each other, and the quasiparticule energy, $\varepsilon_{\mathbf{p}\sigma}$, depends on the entire quasiparticule distribution function, $n_{\mathbf{p}\sigma}$. The interacting produces a variation of the quasiparticule energy

$$\delta\varepsilon_{\mathbf{p}\sigma} = \frac{1}{V} \sum_{\mathbf{p}'\sigma' \neq \mathbf{p}\sigma} f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} \delta n_{\mathbf{p}'\sigma'} \tag{19}$$

so that f can be identified with a second variation of the total energy with respect to changes in the quasiparticule occupation

$$f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} = V \frac{\delta^2 E}{\delta n_{\mathbf{p}\sigma} \delta n_{\mathbf{p}'\sigma'}} \tag{20}$$

We have thus obtained a fundamental energy functional

$$E\{\delta n_{\mathbf{k}\sigma}\} = E_0 + \sum_{k\sigma} \varepsilon^0_{\mathbf{k}\sigma} \delta n_{\mathbf{k}\sigma} + \frac{1}{2V} \sum_{\mathbf{k}\sigma,\mathbf{k}'\sigma'} f_{\mathbf{k}\sigma,\mathbf{k}'\sigma'} \delta n_{\mathbf{k}\sigma} \delta n_{\mathbf{k}'\sigma'} + \ldots \tag{21}$$

where we expect that only momentum close to p_F are relavant at low temperatures. In general the quasiparticule interaction f is split in symmetric and antisymmetric part

$$f_{\mathbf{p}\uparrow,\mathbf{p}'\uparrow} = f_{\mathbf{p}\downarrow,\mathbf{p}'\downarrow} = f^s_{\mathbf{p}\mathbf{p}'} + f^a_{\mathbf{p}\mathbf{p}'} \tag{22}$$

$$f_{\mathbf{p}\uparrow,\mathbf{p}'\downarrow} = f_{\mathbf{p}\downarrow,\mathbf{p}'\uparrow} = f^s_{\mathbf{p}\mathbf{p}'} - f^a_{\mathbf{p}\mathbf{p}'} \tag{23}$$

and, since we restrict to momenta arbitrarily close to the Fermi surface, we expand the interaction in the angle between $\mathbf{p}$ and $\mathbf{p}'$

$$f^s_{\mathbf{p}\mathbf{p}'} = \sum_{l=0}^{\infty} f^s_l P_l(\cos\theta) \tag{24}$$

$$f^a_{\mathbf{p}\mathbf{p}'} = \sum_{l=0}^{\infty} f^a_l P_l(\cos\theta) \tag{25}$$

The conventional Landau parameters are just these interactions multiplied by the density of states

$$F^{s/a}_l = N(0) f^{s/a}_l \tag{26}$$

Specific heat. The specific heat at constant volume is given by

$$C_V = \frac{1}{V} \left.\frac{\partial E}{\partial T}\right|_V \tag{27}$$

The change in temperature induces a change in the occupation numbers, so that we get

$$C_V = \frac{1}{V} \sum_{\mathbf{p}\sigma} \varepsilon^0_{\mathbf{p}\sigma} \frac{\partial n_{\mathbf{p}\sigma}}{\partial T} \tag{28}$$

$$= -\frac{1}{V} \sum_{\mathbf{p}\sigma} \varepsilon^0_{\mathbf{p}\sigma} \frac{\varepsilon_{\mathbf{p}\sigma} - \mu}{T} \frac{\partial n_{\mathbf{p}\sigma}}{\partial \varepsilon_{\mathbf{p}\sigma}} \tag{29}$$

Now, in the limit $T \to 0$, we have

$$\frac{\partial n_{\mathbf{p}\sigma}}{\partial \varepsilon_{\mathbf{p}\sigma}} \to -\delta(\varepsilon_{\mathbf{p}\sigma} - \mu) - \frac{\pi^2 T^2}{6} \frac{\partial^2}{\partial \varepsilon^2_{\mathbf{p}\sigma}} \delta(\varepsilon_{\mathbf{p}\sigma} - \mu) + \mathcal{O}(T^2) \tag{30}$$

and we get

$$C_V \;=\; \frac{N(0)T}{3\pi^2} = \frac{m^* p_F}{3\pi^2} T \tag{31}$$

so that measuring the specific heat at low temperature determone the effective mass.

Effective mass and quasi-particle interaction. In a single component system, there is a simple relation between the effective mass and the Landau parameters of the quasi-particle interaction

$$\frac{m^*}{m} = 1 + \frac{1}{3} F_1^s \tag{32}$$

This relation is intrinsically related to the Galiean invariance of a normal Fermi liquid.

Let us consider an excitation of the system which changes the total momentum per unit volume

$$\delta \mathbf{P} = \frac{1}{V} \sum_{\mathbf{p},\sigma} \mathbf{p} \delta n_{\mathbf{p}\sigma} \tag{33}$$

However, since the number of quasi-particles is equal to the number of real particles, the momentum is also given by the real mass times the group velocity, $\partial \varepsilon_{\mathbf{p}} / \partial \mathbf{p}$, and we have

$$\delta \mathbf{P} = \frac{1}{V} \sum_{\mathbf{p},\sigma} m \frac{\partial \varepsilon_{\mathbf{p}\sigma}\{\delta n_{\mathbf{p}'\sigma'}\}}{\partial \mathbf{p}} \delta n_{\mathbf{p}\sigma} \tag{34}$$

where we have indicated that the quasi-particle energy is a functional of the changes in the occupation number. Using Eq. (21), we have

$$\varepsilon_{\mathbf{p}\sigma}\{\delta n_{\mathbf{p}'\sigma'}\} \;=\; \frac{\delta E\{\delta n_{\mathbf{p}'\sigma'}\}}{\delta n_{\mathbf{p}\sigma}} = \varepsilon^0_{\mathbf{p}\sigma} + \frac{1}{V} \sum_{\mathbf{p}'\sigma'} f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} \delta n_{\mathbf{p}'\sigma'} \tag{35}$$

and, equating Eq. (33) and Eq. (34), we have

$$\frac{\mathbf{p}}{m} \;=\; \frac{\partial \varepsilon^0_{\mathbf{p}\sigma}}{\partial \mathbf{p}} + \frac{1}{V} \sum_{\mathbf{p}'\sigma'} \frac{\partial f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'}}{\partial \mathbf{p}} \delta n_{\mathbf{p}'\sigma'} \tag{36}$$

$$\;=\; \frac{\partial \varepsilon^0_{\mathbf{p}\sigma}}{\partial \mathbf{p}} + \frac{1}{V} \sum_{\mathbf{p}'\sigma'} \frac{\partial f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'}}{\partial \mathbf{p}'} \delta n_{\mathbf{p}'\sigma'} \tag{37}$$

Now, replacing the sum by an integral on the rhs, we can integrate by parts, and use the zero-temperature approximation

$$\frac{\partial \delta n_{\mathbf{p}'\sigma'}}{\partial \mathbf{p}'} = -\frac{\mathbf{p}'}{p_F} \delta(p_F - p') \tag{38}$$

together with $|\mathbf{p}| \to p_F$, we get

$$\frac{\mathbf{p}}{m} = \frac{\mathbf{p}}{m^*} + \frac{1}{p_F} \sum_{\sigma'} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \mathbf{p}' f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} \delta(p_F - |\mathbf{p}'|) \tag{39}$$

Taking the scalar product with $\mathbf{p}$ on both sides, the integral only depends on the angle θ between $\mathbf{p}$ and $\mathbf{p}'$. Using the exansion in Legendre polynoms of the Landau parameter, we finally get

$$\frac{1}{m} \;=\; \frac{1}{m^*} + \frac{p_F}{\pi^2} \int \frac{d\cos\theta}{2\pi} \cos(\theta) \sum_l f_l^s P_l(\cos\theta) \tag{40}$$

$$\;=\; \frac{1}{m^*} \left(1 + \frac{N(0)}{3} f_1 \right) \tag{41}$$

which gives Eq. (32).

Galilean invariance. If we observe the system from a different frame (denoted by a prime) moving with a velocity $\mathbf{u}$, the total energy and the total momentum in the primed frame is related to that of the lab frame by

$$\mathbf{P}' = \mathbf{P} - M\mathbf{u} \tag{42}$$

$$E' = E - \mathbf{P}\cdot\mathbf{u} + \frac{1}{2}Mu^2 \tag{43}$$

Introducing a quasiparticle of momentum $\mathbf{p}$ and energy $\varepsilon_{\mathbf{p}}$ in the lab frame, the momentum in the primed frame increases by $\mathbf{p} - m\mathbf{u}$ while the energy increases by $\varepsilon_{\mathbf{p}} - \mathbf{p}\cdot\mathbf{u} + mu^2/2$,

$$\varepsilon'_{\mathbf{p}-m\mathbf{u}} = \varepsilon_{\mathbf{p}} - \mathbf{p}\cdot\mathbf{u} + \frac{1}{2}mu^2 \tag{44}$$

or

$$\varepsilon'_{\mathbf{p}} = \varepsilon_{\mathbf{p}+m\mathbf{u}} - \mathbf{p}\cdot\mathbf{u} - \frac{1}{2}mu^2 \tag{45}$$

However, in the primed frame, the ground state occupation of the filled Fermi sea is centered around $-m\mathbf{u}$, so that $n'_{\mathbf{p}} = n^0_{\mathbf{p}+m\mathbf{u}}$, and the quasiparticle energy in the primed frame $\varepsilon'_{\mathbf{p}} = \varepsilon_{\mathbf{p}}\{n'_{\mathbf{p}}\}$. Expanding them to first order in u, we obtain

$$\varepsilon'_{\mathbf{p}} = \varepsilon_{\mathbf{p}} + m\sum_{\mathbf{p}'\sigma'}\frac{\delta\varepsilon_{\mathbf{p}}}{\delta n_{\mathbf{p}'}}\frac{\partial n_{\mathbf{p}'}}{\partial \mathbf{p}'}\cdot\mathbf{u} \tag{46}$$

Equating with the Galilean invariance expression, Eq. (45), we get to first order in u,

$$m\frac{\partial\varepsilon_{\mathbf{p}}}{\partial\mathbf{p}}\cdot\mathbf{u} = \mathbf{p}\cdot\mathbf{u} + m\sum_{\mathbf{p}'\sigma'} f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'}\frac{\partial n_{\mathbf{p}'}}{\partial\mathbf{p}'}\cdot\mathbf{u} \tag{47}$$

For simplicity, we can put $\mathbf{u}\sim\mathbf{p}/m$ and we recover the formulas of above.

Non-equilibrium properties. If we are interested in small deviations from equilibrium, considering only long wavelength (low energy) changes, we may characterize the occupation number by a occupation number smoothly varying in time and space, $n_{\mathbf{p}}(\mathbf{r},t)$, with

$$n_{\mathbf{p}}(\mathbf{r},t) = n^0_{\mathbf{p}} + \delta n_{\mathbf{p}}(\mathbf{r},t) \tag{48}$$

which induces a change in the quaiparticle energy

$$\delta\varepsilon_{\mathbf{p}}(\mathbf{r},t) = \sum_{\sigma'\mathbf{p}'}\int d^3\mathbf{r}' f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'}\delta n_{\mathbf{p}}(\mathbf{r},t) \tag{49}$$

In the absence of quasiparticle collisions the time variation of occupation number can be written in the form of a continuity equation

$$\frac{\partial n_{\mathbf{p}}(\mathbf{r},t)}{\partial t} + \nabla_{\mathbf{r}}\cdot[\mathbf{v}_{\mathbf{p}}(\mathbf{r},t)n_{\mathbf{p}}(\mathbf{r},t)] + \nabla_{\mathbf{p}}\cdot[\mathbf{F}_{\mathbf{p}}(\mathbf{r},t)n_{\mathbf{p}}(\mathbf{r},t)] = 0 \tag{50}$$

where the quasiparticle velocity $\mathbf{v}_{\mathbf{p}}(\mathbf{r},t)$ and the force on the quasiparticle $\mathbf{F}_{\mathbf{p}}(\mathbf{r},t)$ is given by the classical Hamilton equations based on the quasiparticle energy

$$\mathbf{v}_{\mathbf{p}}(\mathbf{r},t) = \nabla_{\mathbf{p}}\varepsilon_{\mathbf{p}}(\mathbf{r},t) \tag{51}$$

$$\mathbf{F}_{\mathbf{p}}(\mathbf{r},t) = -\nabla_{\mathbf{r}}\varepsilon_{\mathbf{p}}(\mathbf{r},t) \tag{52}$$

and we have

$$\frac{\partial n_{\mathbf{p}}(\mathbf{r},t)}{\partial t} + \mathbf{v}_{\mathbf{p}}(\mathbf{r},t)\cdot\nabla_{\mathbf{r}}n_{\mathbf{p}}(\mathbf{r},t) + \mathbf{F}_{\mathbf{p}}(\mathbf{r},t)\cdot\nabla_{\mathbf{p}}n_{\mathbf{p}}(\mathbf{r},t) = 0 \tag{53}$$

In the presence of quasiparticle collisions we have to add a collision integral $I[n_{\mathbf{p}}]$ on the rhs, and we get an equation similar to Boltzmann's equation. One can derive the usual hydrodynamic conservation laws.

Zero sound. At low enough temperature, one can argue that the quasiparticle collisions are strongly suppressed, and we can use $I[n] = 0$. We can search for a solution of the hydrodynamic equations which give rise to collective oscillations in the form

$$\delta n_{\mathbf{p}}(\mathbf{r},t) = e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}\phi_{\mathbf{p}} \tag{54}$$

which can be calculated in the linearized regime.

C. Connection with QMC

In order to understand Landau's functional from a microscopic point of view, we recall the ground state wavefunction of the electron gas in the RPA-approximation

$$\Psi_0(\mathbf{R}) \approx N^{-1} \det_{\mathbf{k}i} e^{i\mathbf{k}\cdot\mathbf{r}_i^{\uparrow}} \det_{\mathbf{k}i} e^{i\mathbf{k}\cdot\mathbf{r}_i^{\downarrow}} e^{-\sum_{\mathbf{q}} u_q \rho_{-\mathbf{q}} \rho_{\mathbf{q}}} \tag{55}$$

which is a Slater-Jastrow wavefunction, and should become exact in the non-interacting limit $r_s \to 0$. The normalization factor N will be dropped in the following.

In the ground state, all plane-waves with wavevector $|\mathbf{k}| \leq k_F$ are included in the Slater-determinant. Based on this wavefunction, we can guess possible excitation of the system. Single particle excitations will be characterized by a different occupation of plane-waves in the determinants, $\delta n_{\mathbf{k}\sigma}$, with respect to the ground state occupation. These excitations have exactly the same structure ("quantum number") of the ideal gas excitations. However, we further have collective excitations (plasmons/ phonons), with occupation numbers $a_{\mathbf{q}}$, and the ground state wavefunction is just multiplied by $\rho_{\mathbf{q}}^{a_{\mathbf{q}}}$. Including both types of excitations, we have an explicit functional of the excitations depending on $\delta n_{\mathbf{k}\sigma}$ and $a_{\mathbf{q}}$.

For the electron gas, we saw, that plasmon excitations start with a finite energy, whereas particle-hole (single-particle) excitations can have arbitrary low energies for big enough systems. Therefore, at low enough temperature, we can neglect collective excitations and the low-lying excitations are described by $\delta n_{\mathbf{k}\sigma}$, only. It is then reasonable, to expand the excitation-energies in terms of $\delta n_{\mathbf{k}\sigma}$ and we obtain Landau's functional.

There are several possibilities, that the energies are not any more given as a functional of the occupation numbers. First, of course, when collective excitations mix with single particle states, and both are not any more good "quantum numbers". Second, there can be a problem if two excitations which are in the same symmetry become degenerate. In both cases, the system might not be any more descried by a Fermi liquid.

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