I. BASICS OF STATISTICAL MECHANICS AND QUANTUM MECHANICS

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(Dated: May 18, 2012)

We want to describe basic properties of systems containing a large number N of particles, from mesoscopic ($N \sim 10^3$) up to macroscopic ($N \sim 10^{24}$) systems in d spatial dimensions. (In general we have $d = 3$.) Liquids and solids are typical many-body systems. In many cases, the atoms/ molecules of the system are very well described by classical dynamics. Only at very low temperatures, the quantum nature of the particles becomes important. Liquid helium has been for long time "the example" of a quantum liquid, until experiments on alcaline gases have archieved the neccessary densities/ temperatures to reach quantum degeneracy. Whereas obervation of quantum effects for "real particles" neccessitates quite some experimental efforts, a different kind of quantum liquid is dominating every day's life, at least for condensed matter and solid state physicists: the electron gas.

Many body properties are best described in the framework of statistical mechanics, and we will give a brief introduction in quantum statistics first.

A. Quantum Mechanics

Single partice quantum mechanics. The state of the system is characterized by by a vector $|\Psi\rangle$, and the mean value of an observable with operator $\hat{O}$ in this state is given by

$$\langle \Psi | \hat{O} | \Psi \rangle \tag{1}$$

The state can be representated using any complete basis set, e.g. $\hat{r}|r\rangle = r|r\rangle$ is the position representation

$$\langle \Psi | \hat{O} | \Psi \rangle = \int d\mathbf{r} \int d\mathbf{r}' \Psi^*(\mathbf{r}) \langle \mathbf{r} | \hat{O} | \mathbf{r}' \rangle \Psi(\mathbf{r}) \tag{2}$$

$$\Psi(\mathbf{r}) = \langle \mathbf{r} | \Psi \rangle, \quad \Psi^*(\mathbf{r}) = \langle \Psi | \mathbf{r} \rangle \tag{3}$$

In the Schrödinger eqution the time evolution of the system's state is determined by Schrödinger's equation

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \hat{H} |\Psi\rangle \tag{4}$$

where $\hat{H}$ is the Hamiltonian of the system, and operators are time-independant. Eigenstates of $\hat{H}$ with energiegeladenes E_i are stationary, and form a basis $|E_i\rangle$ with $\hat{H}|E_i\rangle = E_i|E_i\rangle$.

Many particle quantum mechanics: Bosons and Fermions. We can form a basis for many quantum particles by labelling each of them as we do for classical particles. If we have single particle states, e.g. energy eigenstates of a single particle hamiltonian $|E_i\rangle$ we can form a many-body state

$$|\Psi_N\rangle = |E_1\rangle_1 \times |E_2\rangle_2 \times \dots |E_N\rangle_N \tag{5}$$

putting the first particle in an energy state E_1, the second into E_2, etc.. However, there is no possibility of distinguishing quantum particles, e.g. all Hamiltonians known up to now are symmetric with respect to particle permutations. Therefore, all wavefunctions needed to describe the world so far are seperated in two main classes which do not mix: Total symmetric and total antisymmetric wavefunctions

$$|\Psi_N\rangle_{S/A} = \frac{1}{N!} \sum_P (\pm)^{|P|} |E_{P(1)}\rangle_1 \times |E_{P(2)}\rangle_2 \times \dots |E_{P(N)}\rangle_N \tag{6}$$

where the summation is over all possible $N!$ permutations P of particle labels. Particles which are described by symmetrical wavefunctions are called Bosons, antisymmetric wavefunctions describe Fermions. The framework for formal calculations with symmetric/antisymmetric many-body wavefunctions is called *second quantization*.

B. Statistical mechanics

Micro-canonical ensemble. Since any energy eigenstate is equally probably in the micro-canonical ensemble, (N, V, E are constant), the expectation value of an obersable, e.g. the one-body density, $\hat{\rho}^{(1)}(\mathbf{r}) = \sum_i \delta(\mathbf{r} - \hat{\mathbf{r}}_i)$, is given by

$$\langle \hat{\rho}^{(1)} \rangle = \frac{\mathrm{Tr}\left[\delta_{\hat{H},E}\hat{\rho}^{(1)}(\mathbf{r})\right]}{\mathrm{Tr}\left[\delta_{\hat{H},E}\right]} \tag{7}$$

$$= \frac{\sum_i \delta_{E,E_i}\langle E_i|\hat{\rho}^{(1)}(\mathbf{r})|E_i\rangle}{\sum_i \delta_{E,E_i}} \tag{8}$$

where $\hat{H}$ is the Hamiltonian of the system and E_i are its eigenvalues.

The denominator counts just the degeneracy of the energy level under consideration, and it is common to define the entropy S of the system which is proportional to that degeneracy

$$S(E,N,V) = k_B \log \mathrm{Tr}\delta_{\hat{H},E} \tag{9}$$

$$= k_B \log \sum_i \delta_{E,E_i} \tag{10}$$

Canonical ensemble. Let us consider now that the system is connected to a bigger bath and can exchange energy with the bath. Bath and system together form an isolated system which can be described by the micro-canonical ensemble. The probability, p_i, to find the system at energy E_i is equal to the probability that the bath has energy $E - E_i$

$$p_i = \frac{e^{S_B(E-E_i)/k_B}}{\sum_i e^{S_B(E-E_i)/k_B}} \tag{11}$$

where $S_B(E)$ is the entropy of the bath for a given bath-energy E. If the bath is much bigger than the system, we can expand $S_B(E - E_i)$ around $E_i = 0$. We get

$$S_B(E - E_i) = S_B(E) - \frac{dS_B(E)}{dE}E_i + \dots \tag{12}$$

or

$$p_i = \frac{e^{-\beta E_i}}{\sum_i e^{-\beta E_i}} \tag{13}$$

where we have defined the inverse temperature from the thermodynamic relation $1/T = \left.\frac{\partial S}{\partial E}\right|_{V,N}$, and $\beta = 1/k_B T$,. Since the total entropy $S_B(E - E_S) + S(E_S)$ must be maximal for the most likely energy of the system E_s, the temperature of the bath must equal the temperature of the system in equilibrium.

The canonical ensemble describes a system in contact with an energy bath. It is therefore characterized by the temperature T (instead of the energy), and the probability p_α to find the system in a state $|\alpha\rangle$,

$$p_\alpha = \langle\alpha|\hat{\rho}(\beta)|\alpha\rangle \tag{14}$$

The probability is expressed introducing the density operator

$$\hat{\rho}(\beta) = \frac{e^{-\beta\hat{H}}}{Z(\beta)} = \frac{\sum_i |E_i\rangle\langle E_i|e^{-\beta E_i}}{Z(\beta)} \tag{15}$$

The normalisation factor $Z(\beta)$ is called partition function of the canonical ensemble

$$Z(\beta) = \sum_i e^{-\beta E_i}. \tag{16}$$

The mean value, $\langle\hat{O}\rangle$, of some obersable $\hat{O}$ can now be calculated

$$\langle\hat{O}\rangle = \mathrm{Tr}\left[\hat{O}\hat{\rho}(\beta)\right] = \sum_\alpha \langle\alpha|\hat{O}\hat{\rho}(\beta)|\alpha\rangle \tag{17}$$

One important observable is the energy E itself and the mean energy of the system at temperature T writes

$$\langle H\rangle = \frac{\sum_\alpha E_\alpha e^{-\beta E_\alpha}}{\sum_\alpha e^{-\beta E_\alpha}} = -\frac{\partial \log Z(\beta)}{\partial \beta} \tag{18}$$

can be calculated directly from the partition function. The partition function, indeed, plays a central role in statistical mechanics and the connections to thermodynamics is made by introducing the free energy, $F(\beta)$, via

$$Z(\beta) = \exp\left[-\beta F(\beta)\right]. \tag{19}$$

From the thermodynamic definition of the free energy,

$$F = \langle E\rangle - TS \tag{20}$$

we can define the entropy S from thermodynamics. Again, we see that e^S is connected to the typical number of states with energy E in the system.

Classical Statistical Mechanics. The classical limit of the partition function for a system with Hamiltonian

$$\hat{H} = \sum_i \frac{\hat{p}_i^2}{2m} + \sum_i u(\hat{\mathbf{r}}_i) + \sum_{i<j} v(\hat{r}_{ij}) \tag{21}$$

can be obtained by assuming that the momentum and position operators commute. In this limit we have

$$e^{-\beta\hat{H}} \simeq e^{-\beta\sum_i \frac{\hat{p}_i^2}{2m}} e^{-\beta\left[\sum_i u(\mathbf{r}_i)+\sum_{i<j} v(r_{ij})\right]} \tag{22}$$

Using position eigenfunctions $|\mathbf{r}_i\rangle$ as basis of indistinguishable particles to evaluate the trace, we get for the partition function

$$\begin{aligned} Z(\beta) &= \frac{1}{N!^2}\sum_P\sum_{P'}(\pm1)^{|P+P'|}\int d\mathbf{r}_1\langle\mathbf{r}_{P(1)}|_1 \times \int d\mathbf{r}_2\langle\mathbf{r}_{P(2)}|_2\cdots\int d\mathbf{r}_N\langle\mathbf{r}_{P(N)}|_N e^{-\beta\sum_i\frac{\hat{p}_i^2}{2m}} \\ &\quad \times e^{-\beta\left[\sum_i u(\hat{\mathbf{r}}_i)+\sum_{i<j} v(\hat{r}_{ij})\right]}|\mathbf{r}_{P'(1)}\rangle_1 \times |\mathbf{r}_{P'(2)}\rangle_2 \times \cdots |\mathbf{r}_{P'(N)}\rangle_N & (23) \\ &= \frac{1}{N!}\sum_P(\pm1)^{|P|}\int d\mathbf{r}_1\langle\mathbf{r}_{P(1)}|_1 e^{-\beta\frac{\hat{p}_1^2}{2m}}|\mathbf{r}_1\rangle_1 \times \int d\mathbf{r}_2\langle\mathbf{r}_{P(2)}|_2 e^{-\beta\frac{\hat{p}_2^2}{2m}}|\mathbf{r}_2\rangle_2\cdots\int d\mathbf{r}_N\langle\mathbf{r}_{P(N)}|_N e^{-\beta\frac{\hat{p}_N^2}{2m}}|\mathbf{r}_N\rangle_N \\ &\quad \times e^{-\beta\left[\sum_i u(\mathbf{r}_i)+\sum_{i<j} v(r_{ij})\right]} & (24) \end{aligned}$$

Using

$$\langle\mathbf{r}|\mathbf{p}\rangle = (2\pi\hbar)^{-d/2}e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar} \tag{25}$$

we have

$$\langle\mathbf{r}'|e^{-\beta\frac{\hat{p}^2}{2m}}|\mathbf{r}\rangle = \int d\mathbf{p}\langle\mathbf{r}'|\mathbf{p}\rangle e^{-\beta\frac{p^2}{2m}}\langle\mathbf{p}|\mathbf{r}\rangle = \int\frac{d\mathbf{p}}{(2\pi\hbar)^d}e^{-\beta\frac{p^2}{2m}}e^{-i\mathbf{p}(\mathbf{r}-\mathbf{r}')/\hbar} = \int\frac{d\mathbf{p}}{(2\pi\hbar)^d}e^{-\beta\frac{p^2}{2m}}e^{-2m(\mathbf{r}-\mathbf{r}')^2/\hbar^2\beta} \tag{26}$$

and we get

$$Z(\beta) = \frac{1}{N!\hbar^{dN}}\sum_P(\pm1)^{|P|}\int d\mathbf{R}\int\frac{d\mathbf{P}}{(2\pi)^{dN}}e^{-\beta\sum_i p_i^2/2m}e^{-2m\sum_i(\mathbf{r}_{P(i)}-\mathbf{r}_i)^2/\hbar^2\beta}e^{-\beta\left[\sum_i u(\mathbf{r}_i)+\sum_{i<j} v(r_{ij})\right]} \tag{27}$$

As long as the mean distance $\sim n^{-1/d}$ between particle satisfies $n^{-1/d} \gg \sim \lambda_T$ where $\lambda_T = (2\pi/mt)^{1/2}$ is the thermal wave length, all permutations different from the identity will be exponentially suppressed, and we obtain the classical partition function

$$Z(\beta) = \frac{1}{N!\hbar^{dN}}\int\frac{d\mathbf{P}}{(2\pi)^{dN}}\int d\mathbf{R}e^{-\beta E(\mathbf{R},\mathbf{P})} \tag{28}$$

where the energy writes

$$E(\mathbf{R},\mathbf{P}) = \sum_{i=1}^N\frac{p_i^2}{2m} + \sum_{i=1}^N u(r_i) + \sum_{i<j}^N v(r_{ij}) \tag{29}$$

The coordinates $\mathbf{R} \equiv (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$ and momenta $\mathbf{P} \equiv (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ define the phase space of the classical system, and the probability of finding the particles at the point $(\mathbf{R}, \mathbf{P})$ is given by

$$\rho(\mathbf{R}, \mathbf{P}) d\mathbf{R} d\mathbf{P} = \frac{e^{-\beta E(\mathbf{R},\mathbf{P})} d\mathbf{R} d\mathbf{P}}{(2\pi\hbar)^{dN} Z(\beta)} \tag{30}$$

where $d\mathbf{R} \equiv \prod_i d\mathbf{r}_i$. The momentum integration in the partition function

$$Z(\beta) = \frac{1}{N!(2\pi\hbar)^{dN}} \int d\mathbf{R} \int d\mathbf{P} e^{-\beta E(\mathbf{R},\mathbf{P})} \tag{31}$$

can be done explicitly in d spatial dimensions,

$$Z(\beta) = \left(\frac{m}{2\pi\hbar^2 \beta} \right)^{dN/2} \frac{1}{N!} \int d\mathbf{R} e^{-\beta[U(\mathbf{R}) + V(\mathbf{R})]} \tag{32}$$

and only the configuration integral over the potential energy in position space remains, $U(\mathbf{R}) \equiv \sum_i u(r_i)$ and $V(\mathbf{R}) \equiv \sum_{i<j} v(r_{ij})$

It is an important feature of a classical system, that kinetic and potential energy separates in the partition function. Independantly of strength and form of the interactions, the distribution of momenta is always gaussian

$$\rho(\mathbf{P}) d\mathbf{P} \sim \int \rho(\mathbf{R}, \mathbf{P}) d\mathbf{R} d\mathbf{P} \sim e^{-\beta \sum_i p_i^2/2m} \tag{33}$$

and the kinetic energy is a direct measure of the temperature

$$\frac{d}{2} k_B T = \langle \frac{1}{N} \sum_i \frac{p_i^2}{2m} \rangle = \langle \frac{p_1^2}{2m} \rangle \tag{34}$$

This result is the equipartition theorem for classical particles. This relation allows us to determine the temperature in a Molecular Dynamics simulation of in experiments.

Galilean invariance. Let us consider a translational invariant system of interacting particles, e.g. particles on a large ring without explicit single body confining potential, so that the total momentum, $\mathbf{P} = \sum \mathbf{p}_i$, is conserved,

$$H = \frac{\mathbf{P}^2}{2M} + \widetilde{H} \tag{35}$$

where $\widetilde{H}$ is independent of the center-of mass motion, and $M = Nm$ is the total mass. If we observe the system from a different frame (denoted by a prime) moving with a velocity $\mathbf{u}$, the total energy and the total momentum in the primed frame is related to that of the lab frame by

$$\mathbf{P}' = \mathbf{P} - M\mathbf{u} \tag{36}$$

$$H' = H - \mathbf{P} \cdot \mathbf{u} + \frac{1}{2} M u^2 = \frac{[\mathbf{P} - Mu]^2}{2M} + \widetilde{H} \tag{37}$$

If the system is stationary in the primed frame (one has to imagine that the ring contains small impurity potentials which are important to determine the stationary frame, but which are neglected in the calculations), the partition function is given by

$$Z_u = \mathrm{Tr} e^{-\beta H'} \tag{38}$$

If we now consider the total momentum in the lab frame

$$\langle \mathbf{P} \rangle_{\mathbf{u}} \equiv \frac{1}{Z_u} \mathrm{Tr} \left[\mathbf{P} e^{-\beta H'} \right] \tag{39}$$

we can expand for very small velocities $u \to 0$,

$$\langle \mathbf{P} \rangle_{\mathbf{u}} = \beta \langle \mathbf{P} [\mathbf{P} \cdot \mathbf{u}] \rangle_{u=0} + \mathcal{O}(u^2) \tag{40}$$

For a classical system, momenta are always Boltzmann distributed $\propto \exp[-\beta \mathbf{P}^2/2M]$, as we have seen above, and we have $\langle \mathbf{P} \rangle_u = M\mathbf{u}$. In general, quantum systems can show deviations from the classical result. However, as one expects the center of mass motion to become classical for macroscopic systems so that deviations vanish in this limit. As we will see, important deviations from the “normal” behavior may rest even for macroscopic systems, leading to the phenomena of superfluidity and supraconductivity.

C. Ideal Bose gas

Let us consider an ideal Bose gas at low temperatures, and denote the energy eigenstates of energy ε_i of the corresponding single particle Hamiltonian, h, by $|\varepsilon_i\rangle$. Since the particles do not interact, any state of the form

$$|1:\varepsilon_{i_1}\rangle \otimes |2:\varepsilon_{i_2}\rangle \otimes \cdots \otimes |N:\varepsilon_{i_N}\rangle \tag{41}$$

is an eigenstate of total energy $\sum_j \varepsilon_{i_j}$ of the N-particle Hamiltonian $H = \sum_i h_i$. However, whereas for (ficticious) quantum particles obeying Boltzmann statistics,

$$|1:\varepsilon_{i_2}\rangle \otimes |2:\varepsilon_{i_1}\rangle \otimes \cdots \otimes |N:\varepsilon_{i_N}\rangle \tag{42}$$

is a new eigenstate, degenerate in energy with the previous, the symmetric states obtained from both are identical. All possible bosonic eigenstates of the Hamiltonian are therefore simply labelled by the occupation number n_i of the single-particle eigenstate $|\varepsilon_i\rangle$, whereas Boltzmann statistics would also require the actual particle number.

For an ideal Bose gas the total energy of the system is then given by a summation of the occupation number n_i of each energy eigenstate of energy ε_i

$$E(\{n_i\}) = \sum_i n_i \varepsilon_i \tag{43}$$

and the partition function at temperature $T = 1/\beta$ in the canonical ensemble writes

$$Z_N = {\sum_{\{n_i\}}}' e^{-\beta E(\{n_i\})} \tag{44}$$

where the summation over all occupation numbers is done with the constraint that they sum up to the total number of Bosons N,

$${\sum_{\{n_i\}}}' = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots \delta_{\sum n_i, N} \tag{45}$$

Going over to the grand canonical ensemble, we can avoid this constraint introducing the chemical potential μ and we get

$$\begin{aligned} Z &= \sum_N e^{\beta\mu N} Z_N && (46)\\ &= \sum_{n_1=0}^{\infty} e^{-\beta n_1(\varepsilon_1-\mu)} \sum_{n_2=0}^{\infty} e^{-\beta n_2(\varepsilon_2-\mu)} \ldots && (47)\\ &= \prod_i \left[\frac{1}{1-e^{-\beta(\varepsilon_i-\mu)}}\right] && (48)\end{aligned}$$

We get the mean occupation number of mode i by varying ε_i, e.g.,

$$\begin{aligned} N_i &\equiv \langle n_i\rangle = -\frac{\delta \log Z}{\delta \beta\varepsilon_i} && (49)\\ &= \frac{1}{e^{\beta(\varepsilon_i-\mu)}-1} && (50)\end{aligned}$$

which is the familar Bose-Einstein distribution.

Bose Einstein condensation. The number of particles $N = \sum_i \langle n_i\rangle$ is an increasing function for increasing (negative) chemical potential. However, at $\mu = \varepsilon_0$ a singularity appears, where

$$N_0 \simeq \frac{1}{\beta|\epsilon_0-\mu|}, \quad \text{for}\, \mu \to \varepsilon_0 \tag{51}$$

It is important to distinguish if this singularity is approached for a system of finite density at finite temperature so that the singularity indicates a transition to a new phase, e.g. in contrast to a singularity occuring as a consequence of a diverging density. Therefore, let us calculate the total number of particles for a fixed system,

$$N = \sum_{i>0} N_i = \int_{\delta}^{\infty} d\epsilon N(\epsilon) \frac{1}{e^{\beta(\epsilon-(\mu-\varepsilon_0))}-1} \tag{52}$$

where we have introduced the density of states

$$N(\epsilon) = \sum_i \delta(\epsilon - (\varepsilon_i - \varepsilon_0)) \tag{53}$$

and introduced a small offset $\varepsilon_1 - \varepsilon_0 > \delta > 0$. The convergence properties of the integral in Eq. (52) in the limit $\delta \to 0$ determine if the singularity for N_0 is approached for finite parameters of the system. In the case where the density of states is non vanishing for small energies, $N(\epsilon) = c > 0$ for $\epsilon \to 0$, the integral diverges $Nc \sim -T \log \delta/T \gg 1$ for $\mu = \epsilon_0$, whereas for any $N(\epsilon) \sim \epsilon^x$ with $x > 0$, it approaches a finite limiting value N_c, and we may expect a phase transition to occur at finite temperature.

Homogenous system. For a homogeneous system in 3 dimensions, where the energy eigenstates are plane waves with energies $\varepsilon_k = \hbar^2 k^2/2m$, and the density can be expressed by

$$n = \frac{1}{V}\sum N_k = \int \frac{d^3k}{(2\pi)^3}\frac{1}{e^{\beta(\hbar^2k^2/2m-\mu)}-1} = \left(\frac{2m}{\hbar^2\beta}\right)^{3/2}\frac{1}{8\pi^3}\int d^3x \sum_{n=1}^{\infty} e^{-n(x^2-\beta\mu)} = \lambda^{-3}\sum_n \frac{e^{n\beta\mu}}{n^{3/2}} \tag{54}$$

where we have introduced the thermal wavelength $\lambda = \sqrt{2\pi\hbar^2/(mT)}$. Defining

$$g_z(x) = \sum_n n^{-z} x^n \tag{55}$$

we have

$$n\lambda^3 = g_{3/2}(e^{\beta\mu}) \tag{56}$$

Einstein condensation occurs at a finite density/ temperature in three dimensions, since the density at $\mu = 0$ is finite

$$n_c\lambda^3 \simeq 2.61\ldots \tag{57}$$

This is the critical density of Bose Einstein condensation. Note that we approached $\mu \to 0$ such that the ground state occupation density $n_0 = N_0/V$ remains zero.

Below the critical temperature (or above the critical density),

$$\beta\mu \sim -\frac{1}{V} \tag{58}$$

and the ground state density $n_0 \sim 1$. However, whereas the condensate is macroscopically occupied, the excited state occupation remains of order one. Since the density of states is non-vanishing in two or one dimensions, Bose-Einstein condensation (in the sense of a true phase transition in the thermodynamic limit) does not occur in these lower dimensions.

Superfluidity. Let us now discuss the response of the homogeneous d-dimensional Bose gas to a moving system with small velocities. From the discussion above, we need to calculate the total momentum squared, e.g. in the x-direction

$$\begin{aligned}\langle \mathbf{P}_x^2\rangle = \left\langle \left[\sum_{\mathbf{k}} k_x n_{\mathbf{k}}\right]^2\right\rangle &= \sum_{\mathbf{k}} k_x^2\left[\langle n_{\mathbf{k}}^2\rangle - \langle n_{\mathbf{k}}\rangle^2\right] + \sum_{\mathbf{k}}\sum_{\mathbf{p}} k_x p_x \langle n_{\mathbf{k}} n_{\mathbf{p}}\rangle \\ &= \frac{1}{d}\sum_{\mathbf{k}} k^2 \frac{\partial}{\partial(-\beta\varepsilon_k)}\langle n_{\mathbf{k}}\rangle\Big|_{\beta\mu} + \sum_{\mathbf{k}}\sum_{\mathbf{p}} k_x p_x \langle n_{\mathbf{k}}\rangle\langle n_{\mathbf{p}}\rangle\end{aligned} \tag{59}$$

Whereas the last term on the rhs is clearly zero, we now use that $\varepsilon_k = k^2/2m$, to simplify

$$\langle \mathbf{P}_x^2\rangle = -\frac{2m}{d}\sum_{\mathbf{k}} N_k\Big|_{\beta\mu} = -\frac{2mV}{d}\frac{\partial n}{\partial\beta}\Big|_{\beta\mu} = -\frac{2mV}{d}\frac{\partial}{\partial\beta}\lambda^{-d}g_{d/2}(e^{\beta\mu})\Big|_{\beta\mu} = TnVm = TM \tag{60}$$

We see that for $N \le N_c$ the system behaves "normal", howeever, for $N > N_c$ the particles in the condensate have zero momenta and will not contribute to the fluctuations of the total momenta and we have

$$\langle \mathbf{P}\rangle_u = (N - N_0)M\mathbf{u} \tag{61}$$

and the normal mass is smaller than the total mass leading to a the superfluid mass density N_0M/V in the non-interacting system.

Non-ideal gases. Important questions arises concerning what properties of the ideal gas survive when the particles interact. How is Bose condensation and superfluidity affected and what are the changes in dilute gases? How to describe dense fluids, e.g. ^{4}He?

D. Ideal Fermi gas

Let us now discuss the ideal Fermi gas where we have to antisymmetrize the Boltzmann energy-eigenstates. Similar to a Bose system, two Boltzmann states with the same set of occupation numbers but occupied by different permutation of particle labels will describe exactly the same state after antisymmetrization. In addition, we can see that if two or more particles occupy the same state, the state vanishes after antisymmetrization, e.g.

$$[1 - P_{ij}] \, |i : \varepsilon_k\rangle \otimes |j : \varepsilon_k\rangle = 0 \tag{62}$$

Therefore, similar to bosons, the eigenenergies of the ideal system writes

$$E(\{n_i\}) = \sum_i n_i \varepsilon_i \tag{63}$$

but, now, all occupation numbers are either zero or one, $n_i = 0, 1$. It is again advantageous to consider the grand canonical ensemble where we have

$$\begin{aligned} Z &= \sum_N e^{\beta\mu N} Z_N && (64)\\ &= \sum_{n_1=0}^{1} e^{-\beta n_1(\varepsilon_1-\mu)} \sum_{n_2=0}^{1} e^{-\beta n_2(\varepsilon_2-\mu)} \ldots && (65)\\ &= \prod_i \left[1 + e^{-\beta(\varepsilon_i-\mu)}\right] && (66)\end{aligned}$$

We then get the mean occupation number of mode i by varying ε_i, e.g.,

$$\begin{aligned} N_i &\equiv \langle n_i\rangle = -\frac{\delta \log Z}{\delta \beta\varepsilon_i} && (67)\\ &= \frac{1}{e^{\beta(\varepsilon_i-\mu)} + 1} && (68)\end{aligned}$$

which is the Fermi-Dirac distribution. We see that, in contrast to the Bose distribution, no singularity can occur at finite temperature. In the zero temperature limit we have

$$N_i = \theta(\mu - \varepsilon_i) \tag{69}$$

so that all single particle levels with energies less than the chemical potential are occupied.

Appendix: gaussian integrals

Gaussian integrals play a central role in statistical mechanics, in d dimensions we have

$$I_d(\gamma) = \int d^d x e^{-\gamma \mathbf{x}^2} \tag{70}$$

where $\mathbf{x}^2 = x_1^2 + x_2^2 + \ldots + x_d^2$. We can easily relate the integrals beween different dimensions using

$$I_d(\gamma) = [I_1(\gamma)]^d \tag{71}$$

In $d = 2$ dimensions, the integral can be canlculated easily changing to radial variables, $r = \sqrt{x_1^2 + x_2^2}$ and $\cos\theta = x_1/r$,

$$I_2 = \int_{-1}^{1} d\cos\theta \int dr r e^{-\gamma r^2} = \pi \int d(r^2) e^{-\gamma r^2} = \frac{\pi}{\gamma} \tag{72}$$

So we get the general result

$$I_d(\gamma) = \left(\frac{\pi}{\gamma}\right)^{d/2} \tag{73}$$