Condensed Fraction of an Atomic Bose Gas Induced by Critical Correlations

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We study the condensed fraction of a harmonically trapped atomic Bose gas at the critical point predicted by mean-field theory. The nonzero condensed fraction f_0 is induced by critical correlations which increase the transition temperature T_c above T_c^{MF}. Unlike the T_c shift in a trapped gas, f_0 is sensitive only to the critical behavior in the quasiuniform part of the cloud near the trap center. To leading order in the interaction parameter a/λ_0, where a is the s-wave scattering length and λ_0 the thermal wavelength, we expect a universal scaling $f_0 \propto (a/\lambda_0)^4$. We experimentally verify this scaling using a Feshbach resonance to tune a/λ_0. Further, using the local density approximation, we compare our measurements with the universal result obtained from Monte Carlo simulations for a uniform system, and find excellent quantitative agreement.

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Some of the most interesting fundamental problems of many-body physics involve strong interparticle correlations, and cannot be addressed by mean-field (MF) theories. Harmonically trapped ultracold atomic gases are promising candidates for highly controllable "quantum simulation" of such intricate many-body scenarios [1]. However, for testing the existing theories of spatially uniform systems, it is often important to experimentally extract information on local properties of a nonuniform trapped gas (see, e.g., [2–4]).

The effect of interactions on Bose-Einstein condensation of a dilute gas is a classic example of a difficult beyond-MF problem, which has challenged theorists for decades [5–20]. It is also an example of a situation where harmonic confinement both quantitatively and qualitatively alters the physics [21–30]. For a uniform gas the interaction shift of the critical temperature T_c cannot be calculated to any order in the interaction strength using perturbation theory, owing to strong correlations that develop near the critical point. On the other hand, nonuniformity of a trapped atomic gas results in a significant MF shift of T_c (at fixed total atom number) [21]. It also diminishes the relative contribution of the more interesting beyond-MF effects, in essence because near T_c only a small fraction of the cloud is actually in the critical regime (see Fig. 1). Only recently have the beyond-MF effects on condensation of an atomic gas become experimentally accessible [30], and many questions are still open.

In this Letter, we study the condensed fraction (f_0) of an atomic Bose gas at the critical point predicted by MF theory. By definition f_0 vanishes within MF theory, and directly measures the effect of critical correlations which shift T_c above T_c^{MF}. Moreover, while the T_c shift itself strongly depends on the global properties of a nonuniform gas, f_0 measurements directly probe the quasiuniform critical region near the center of the trap. To leading order in the strength of interactions we predict a universal scaling $f_0 \propto (a/\lambda_0)^4$, where $a > 0$ is the s-wave scattering length and λ_0 the thermal wavelength at the ideal-gas critical temperature T_c^0. Using a Feshbach resonance in a ^{39}K gas to tune a/λ_0, and accurately measuring condensed fractions in the range 0.1%–1%, we experimentally verify this prediction. Further, we directly relate our measurements to the universal critical behavior seen in the classical-field Monte Carlo simulations of a *uniform* system [31], and find excellent quantitative agreement.

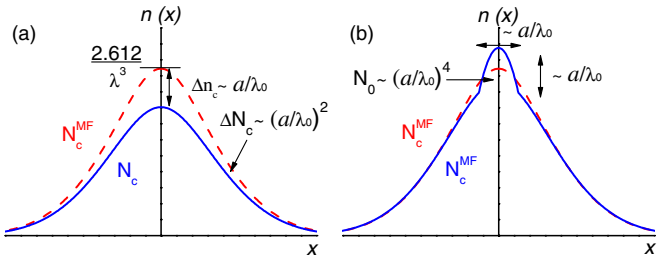

FIG. 1 (color online). Beyond-mean-field effects near the critical point in a harmonically trapped Bose gas. (a) For a fixed temperature, the density distribution at the critical point $N = N_c < N_c^{\mathrm{MF}}$ (solid blue line) is compared with the mean-field prediction (dashed red line). In the trap center we expect $n_c^{\mathrm{MF}} - n_c \propto a/\lambda_0$, characteristic of the critical behavior in a uniform system. However, the experimentally measured N_c shift, $N_c^{\mathrm{MF}} - N_c \propto (a/\lambda_0)^2$, is dominated by the density shift *outside* the central critical region, and is not directly related to the n_c shift. (b) If N is increased to $N_c^{\mathrm{MF}} > N_c$, a small condensate induced by critical correlations forms within the critical region of size $\propto a/\lambda_0$. The condensed atom number $N_0 \propto (a/\lambda_0)^4$ directly relates to the critical density shift $\Delta n_c \propto a/\lambda_0$. (Note that the plots are only indicative of the relevant physics, and are not meant to be quantitatively accurate.)

In Fig. 1(a) we illustrate the qualitative difference between the beyond-MF shifts of the critical point in a uniform and a trapped gas, and in Fig. 1(b) the expected scaling of the condensed fraction at the MF critical point. For visual clarity, here we fix the temperature of the gas and consider the interaction shift of the critical density n_c (in the center of the trap) and the critical atom number N_c. The quadratic beyond-MF N_c shift was observed in [30]. The expected linear n_c shift is characteristic of a uniform system and more interesting from the point of view of the theory of critical behavior. However, it cannot be experimentally verified without a direct probe of the local density in a 3D cloud.

Here we show how to overcome this problem and experimentally obtain information about the critical behavior in a uniform gas. We first outline some general scaling arguments, then present our experimental results, and finally make a quantitative comparison of our measurements with the theory based on the Monte Carlo simulations of Ref. [31].

In a uniform system, ideal-gas condensation occurs at a chemical potential $\mu_c^0 = 0$, and a critical phase space density $n\lambda^3 = \zeta(3/2) \approx 2.612$, where ζ is the Riemann function. In an interacting gas there is no T_c shift at MF level, i.e., $T_c^{\mathrm{MF}} = T_c^0$. To leading order in $a/\lambda_0 \ll 1$ the expected beyond-MF T_c shift is given by [7–20]

$$\frac{\Delta T_c}{T_c^0} \approx c\frac{a}{\lambda_0}, \tag{1}$$

where $\Delta T_c = T_c - T_c^0$, and $c \approx 1.8$ [13,14]. Equivalently, the n_c shift at constant T is $\Delta n_c/n_c^0 \approx -(3/2)\Delta T_c/T_c^0$.

An important point is that, at both MF and beyond-MF level, the interactions differently affect T_c (or equivalently n_c) and the critical chemical potential μ_c. The simple MF shift $\beta\mu_c^{\mathrm{MF}} = 4\zeta(3/2)a/\lambda_0$, where $\beta = 1/k_{\mathrm{B}}T$, has no effect on condensation, and to lowest beyond-MF order [32]:

$$\beta\mu_c \approx \beta\mu_c^{\mathrm{MF}} + B_2\left(\frac{a}{\lambda_0}\right)^2. \tag{2}$$

The qualitative difference between Eqs. (1) and (2) highlights the fact that the problem of the T_c shift is nonperturbative and near criticality the equation of state does not have a regular expansion in μ (otherwise one would get $\Delta n_c \propto \mu_c^{MF} - \mu_c$).

In a harmonically trapped gas T_c is defined for a given atom number N, rather than for a given density n. For an ideal gas $k_{\mathrm{B}}T_c^0 = \hbar\omega[N/\zeta(3)]^{1/3}$, where $\zeta(3) \approx 1.202$. Within the local density approximation (LDA) one expects the uniform-system results for n_c and μ_c to apply in the center of the trap, $\mathbf{r} = 0$. Elsewhere in the trap the local chemical potential is $\mu(\mathbf{r}) = \mu(0) - m\omega^2 r^2/2$, where m is the atom mass and ω the trapping frequency. The result for the T_c shift, however, does not carry over so easily to the nonuniform case. At T_c, the size of the central critical region in a trapped cloud is $r_c \sim (a/\lambda_0)R_T$, where $R_T = \sqrt{k_{\mathrm{B}}T/m\omega^2}$ is the thermal radius [25]. Combining this with $\Delta n_c \sim a/\lambda_0$, one obtains a very small beyond-MF shift of the critical number of atoms within the critical region, of the order $(a/\lambda_0)^4$. However, the actual beyond-MF N_c shift is instead $\propto (a/\lambda_0)^2$. In fact, the experimentally observed T_c shift qualitatively mirrors Eq. (2):

$$\frac{\Delta T_c}{T_c^0} \approx b_1\frac{a}{\lambda_0} + b_2\left(\frac{a}{\lambda_0}\right)^2. \tag{3}$$

Here $b_1 \approx -3.426$ is an analytical, strictly MF result [21], and $b_2 = 46 \pm 5$ was measured in [30].

We can qualitatively understand the similarity of Eqs. (2) and (3) by noting that (i) the interaction shift of μ_c affects the density everywhere in the trap, (ii) away from the critical point the equation of state is regular in μ and the local density shift is simply proportional to μ_c, at both MF and beyond-MF level [33], and (iii) the contribution to N_c from the noncritical region outweighs the contribution from within the critical region by a large factor $\sim(\lambda_0/a)^3$.

To conclude this analysis, the beyond-MF T_c shift observed in a trapped gas [30] is directly related to the beyond-MF μ_c shift (in either trapped or uniform system). It does not, however, directly reveal the expected linear n_c shift and the theoretically most intriguing nonperturbative connection between μ_c and n_c shifts.

By studying the condensed fraction f_0 at the MF-predicted critical point we overcome the problem of the absence of the local density probe. Simply put, instead of asking how N_c is reduced with respect to N_c^{MF} by critical correlations, we ask how many atoms pile up in the condensate if (at constant T) we increase the total atom number to $N_c^{\mathrm{MF}} > N_c$. Experimentally, the obvious advantage is that the condensed and thermal component can be clearly distinguished in standard time-of-flight (TOF) expansion, thus allowing us to use a "global" measurement technique to access the local behavior of the gas within the critical region. Theoretically, the analogous quantity for a uniform gas, n_0/n (where n_0 is the condensate density), was first considered by Holzmann and Baym [34]. Although the formal proof is rather involved, the main scaling result is intuitive, $n_0/n \propto \Delta n_c \propto a/\lambda_0$ [35]. From this result we immediately obtain $f_0 \propto (a/\lambda_0)^4$, as illustrated in Fig. 1(b). The harmonic trapping potential still affects the scaling of f_0 with a/λ_0, but in this case the results for a harmonic and a uniform system are trivially related by the volume of the critical region, $\propto (a/\lambda_0)^3$.

To experimentally measure f_0 we use an optically trapped cloud of ^{39}K atoms in the $|F, m_F\rangle = |1, 1\rangle$ hyperfine state, in which the strength of interactions can be tuned via a Feshbach resonance centered at 402.5 G [36]. Our setup and measurements close to the critical point are described in [30,37]. Briefly, we prepare partially condensed clouds at various values of the scattering length

a, and then let the number of atoms in the trap gradually decay through inelastic processes, while finite trap depth and sufficiently high rate of elastic collisions ensure that the sample remains in equilibrium at an approximately constant temperature. For the measurements presented here, $N \approx (4-5)\times 10^5$, the geometric mean of the trapping frequencies in our nearly isotropic trap is $\bar{\omega}/2\pi \approx 80$ Hz, and $T \approx 250$ nK, corresponding to $\lambda_0 \approx 10^4 a_0$, where a_0 is the Bohr radius. To discern condensed fractions as low as $\sim 0.1\%$ we switch a close to zero during TOF, thus minimizing the condensate expansion [30].

Our experimental results are summarized in Fig. 2. Starting at zero for small a (in agreement with MF theory), f_0 grows to $\sim 1\%$ at $a \approx 350 a_0$. At even larger a the unfavorable ratio of the three-body loss rate to the two-body elastic collision rate precludes reliable equilibrium measurements [30].

Because it is essential that we measure f_0 at $N = N_c^{\mathrm{MF}}$, we explain in detail how we ensure this, and assess the errors in our measurements and analysis. Absolute N calibration is usually limited by the uncertainty in the absorption-imaging cross section σ. N_c^{MF} can be corrected for small effects such as the finite-size T_c shift [38] (in our case $\approx 1\%$), but its absolute accuracy is limited by the experimental uncertainty in $\bar{\omega}$ (in our case $\approx 2\%$). To eliminate these systematic errors to leading order, for every "measurement" series at a given a (data point in Fig. 2) we also take a "reference" series at $a \approx 50 a_0$. In the weakly interacting gas with $a/\lambda_0 \approx 0.005$ the beyond-MF effects are negligible (the expected f_0 is $<10^{-5}$) and the critical atom number N_c, obtained for $N_0 \to 0$, is experimentally indistinguishable from N_c^{MF} [30]. Setting $N_c^{\mathrm{MF}} = N_c$ for the reference series therefore introduces a negligible error,

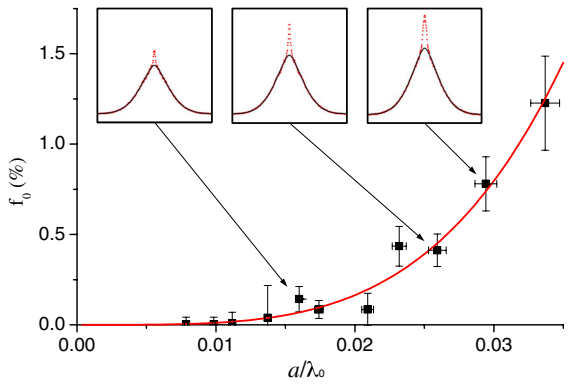

FIG. 2 (color online). Condensed fraction of an atomic gas induced by critical correlations. The condensed fraction f_0 is measured at the point where the total atom number is $N = N_c^{\mathrm{MF}} > N_c$. A fit to the data (solid line) with the function $f_0 \propto (a/\lambda_0)^x$ gives an exponent $x = 3.9 \pm 0.4$, in agreement with the predicted $x = 4$. Vertical error bars are statistical and horizontal error bars reflect the 0.1 G uncertainty in the position of the Feshbach resonance. The insets show representative column density distributions after 19 ms TOF.

and allows us to calculate N_c^{MF} for the measurement series. Since the two series have identical σ and $\bar{\omega}$ (hence also equal finite-size shifts) [39], we only need the *ratio* of the two N_c^{MF} values, given by the standard MF theory (see also [33]). From our reference series we can also absolutely calibrate σ and all atom numbers to $\approx 6\%$, but we do not rely on this calibration and this uncertainty does not affect our results. Our cancellation of systematic errors is valid only to first order; e.g., we assume that the small MF and finite-size T_c shifts are additive. However, higher order corrections are much smaller than our statistical errors, coming from shot-to-shot variations in atom numbers and temperatures. Combining N and T fluctuations, our statistical error in N_c^{MF} is typically $\approx 1\%$. For comparison, the difference between N_c^{MF} and N_c is as large as 10% in our most strongly interacting samples [40].

We fit our f_0 data with a function $(a/\lambda_0)^x$ where x is a free parameter. The fit yields $x = 3.9 \pm 0.4$, in agreement with the predicted $x = 4$. This confirmation of the expected scaling of f_0 with a/λ_0 is the first main result of this Letter.

We now quantitatively relate our measurements to Monte Carlo (MC) calculations for a uniform gas. Following [31] we first define the reduced chemical potential [42]

$$X = \frac{\mu - \mu_c}{32\pi^3 (a/\lambda_0)^2 k_B T}. \tag{4}$$

Next, following [13] we calculate X_0, the value of X in the center of the trap for $N = N_c^{\mathrm{MF}}$. [Because of logarithmic corrections this is a slightly different condition from $\mu(0) = \mu_c^{\mathrm{MF}}$, but this distinction is not experimentally observable.] We use the experimental value $b_2 = 42 \pm 2$ and $b_2^{\mathrm{MF}} = 11.7 \pm 0.1$ [30,33] to get

$$X_0 \approx \frac{3\zeta(3)}{32\pi^3 \zeta(2)} (b_2 - b_2^{\mathrm{MF}}) = 0.067 \pm 0.005. \tag{5}$$

For a uniform system the reduced condensate density $\tilde{f}(X)$, defined by $n_0 \lambda_0^3 = 16\pi^3 (a/\lambda_0) \tilde{f}(X)$, was tabulated in [31] using MC simulations. Invoking LDA, for a harmonically trapped gas we get

$$\frac{N_0}{N_c^0} = \frac{\sqrt{2}(4\pi)^7}{4\zeta(3)} \left(\frac{a}{\lambda_0}\right)^4 \int_0^{X_0} \tilde{f}(X) \sqrt{X_0 - X}\, dX. \tag{6}$$

Writing $(N_0/N_c^0)^{1/4} = \alpha(a/\lambda_0)$ and numerically evaluating the integral in Eq. (6), using the results of [31], we get the numerical coefficient $\alpha_{\mathrm{MC}} = 10.4 \pm 0.4$.

In Eq. (6) N_0 is calculated at $N = N_c^{\mathrm{MF}}$ but normalized to N_c^0. This expression therefore differs from f_0 by a factor of N_c^{MF}/N_c^0. This difference does not affect the leading $(a/\lambda_0)^4$ term and is relevant only at the $(a/\lambda_0)^5$ level. Nevertheless, for a direct quantitative comparison, in Fig. 3 we normalize the measured N_0 values to N_c^0, and assume the quartic dependence on a/λ_0. The linear fit to

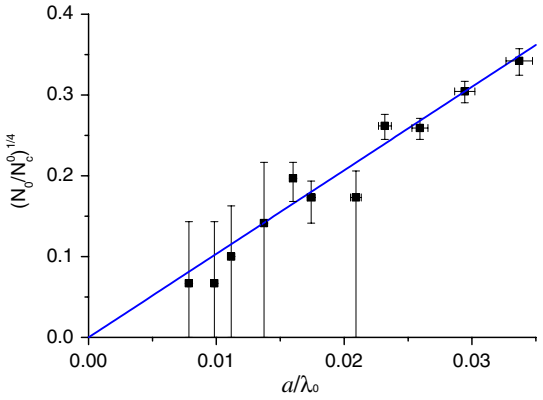

FIG. 3 (color online). Comparison with Monte Carlo calculations for a uniform system. To quantitatively compare our data with the MC simulations we plot $(N_0/N_c^0)^{1/4}$ versus a/λ_0 (see text). A linear fit gives a gradient of $\alpha_{\mathrm{exp}} = 10.3 \pm 0.3$, in excellent agreement with the prediction $\alpha_{\mathrm{MC}} = 10.4 \pm 0.4$. The error bars are obtained using the limiting values from Fig. 2; the points with large error bars do not significantly affect the fitted value of α_{exp} but are clearly consistent with it.

$(N_0/N_c^0)^{1/4}$ yields the experimental value $\alpha_{\mathrm{exp}} = 10.3 \pm 0.3$, in excellent agreement with the Monte Carlo result.

For another comparison, it is interesting to convert X_0 into N_0 using the standard Thomas-Fermi (TF) law. This MF law is valid well below T_c, where $N_0 \approx N$, but should not hold close to the critical point. For a given X_0, the TF law also predicts $N_0 \propto (a/\lambda_0)^4$. However, it corresponds to $\tilde{f}(X) = X$ and gives $\alpha_{\mathrm{TF}} = 8.2 \pm 0.4$. This result underestimates the condensed fraction f_0 by a factor $(\alpha_{\mathrm{MC}}/\alpha_{\mathrm{TF}})^4 \approx 2.6$, and we experimentally exclude it at about 4 sigma level. This confirms that near T_c mean-field theory fails on both sides of the critical point.

In conclusion, we have studied the condensed fraction of an atomic gas induced by interparticle correlations at a point where no condensate is predicted by mean-field theory. Building on the recent observation of correlation effects on the condensation temperature of a trapped gas, this work makes a more direct connection with the critical behavior in a homogeneous system. We experimentally confirm the predicted scaling $f_0 \propto (a/\lambda_0)^4$, which highlights the conceptual difference between the interaction shifts of the critical density (characteristic of a uniform system) and the critical atom number in a harmonically confined cloud. Moreover, we demonstrate excellent quantitative agreement between our experiments and Monte Carlo simulations for a homogeneous gas. In a more general context, this provides an example of the potential of ultracold atomic gases for quantitative quantum simulation of intricate beyond-mean-field phenomena in uniform many-body systems.

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[32] Note that B_2 is not just a constant but contains logarithmic corrections in a/λ_0 [13]. We neglect these in our discussion since they are not discernible at the current level of experimental precision.
[33] For quantitative comparison of theory and experiment we take into account that (i) b_2 includes a small MF contribution, $b_2^{\mathrm{MF}} = 11.7 \pm 0.1$, which we obtain by a direct self-consistent numerical simulation, and (ii) by fitting the data of [30] with the linear T_c shift fixed to $b_1 = -3.426$ we improve the experimental estimate for the quadratic shift to $b_2 = 42 \pm 2$. These corrections do not affect our general scaling arguments.

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[35] Strictly speaking this linear scaling holds for any fixed distance from the critical point given by $(\mu - \mu_c) \times (\lambda_0/a)^2 = \text{const}$ [34]. By applying it to the MF critical point we again neglect the theoretically expected logarithmic corrections to $\mu_c^{\text{MF}} - \mu_c$, which are so far not experimentally observable.
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[39] The two data series are taken concurrently, by alternating experimental shots, in order to guard against any small long-term experimental drifts in either σ or $\bar{\omega}$.
[40] The f_0 values are much smaller than 10% because of the nonsaturation of the thermal component [41]. When N is increased to $N_c^{\text{MF}} > N_c$, most of the extra particles, $N_c^{\text{MF}} - N_c$, actually go into the nonsaturated thermal cloud.
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[42] Note that the denominator $32\pi^3(a/\lambda_0)^2 k_B T$ in Eq. (4) gives the natural correlation energy scale, which must be much larger than $\hbar\bar{\omega}$ for correlation effects to be observable in a finite-size system. This inequality is well satisfied for all a/λ_0 where the measured f_0 is clearly distinguishable from zero. On the other hand the two energy scales are essentially equal at $a/\lambda_0 = 0.005$, in agreement with the assumption that at that point the correlation effects are negligible.