Anderson Localization
1958-2008

From electrons, towards ultrasound
and … cold atoms

Bart van Tiggelen
Laboratoire de Physique et Modélisation des Milieux Condensés
Université Joseph Fourier /CNRS

LPMC
laboratoire
de physique et
de modélisation
des milieux condensés

Scattering Systems with complex Dynamics FOR760 2008: October 6 2008

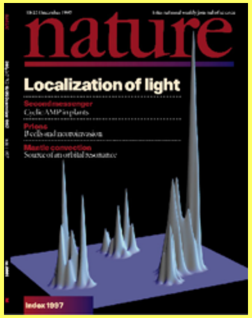
Localization [..] very few believed it at the time, and even fewer saw its importance, among those who failed was certainly its author.
It has yet to receive adequate mathematical treatment, and one has to resort to the indignity of numerical simulations to settle even the simplest questions about it.

P.W. Anderson, Nobel lecture, 1977

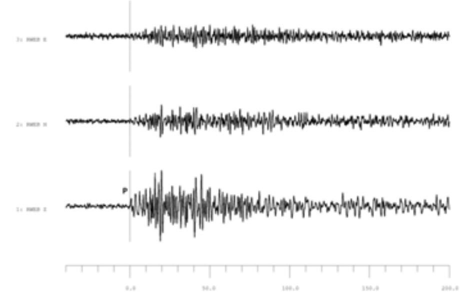
Half a century…….

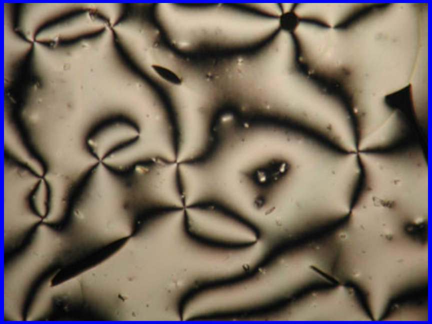
1958	Anderson	« vanishing of diffusion »
1960	Mott/Ioffe-Regel	$\ell \le \frac{\lambda}{2\pi}$
1965	Mott	Minimum conductivity Variable range hopping
1972	Thouless	Sensitivity to BC: g < 1 (Thouless criterion)
1977	Anderson/Mott	Nobel Prize
1980	« gang of four »	Scaling theory $\frac{\partial \log(g)}{\partial \log(L)}$
1980	Götze, Vollhardt, Wölfle	Self-consistent theory
1982	Halperin, Pruisken	Scaling theory of Quantum Hall effect
>1982	Sharvin, Lagendijk, Maret, Maynard,…	Weak localization Mesoscopic physics!

1983	Fröhlich & Spencer	Mathematical proof for 3D Anderson model
1984	Anderson	25 years localization « *unrecognizable monster* »
1986	Anderson	« Theory of white paint »
1986	Kramer, Mackinnon, Economou, Soukoulis, Schreiber	Tight binding model
1987	Papanicolaou, Sheng	Prediction of Localization of Seismic Waves in layered Earth Crust
1987	Souillard	Localization of Gravitational Waves in Universe?
1988	John	Prediction of Localization of light in Photonic crystals
1990	Dorokhorov, Mello etal, Beenakker Altschuler	DMPK equation for wire interactions

> 1991	Bell-labs Weaver Genack Exxon (Sheng etal)	2D localization microwaves 2D localization of ultrasound Q1D localization microwaves 2D localization of bending waves
>1995	BEC community	Localization of light in BEC cold atomes gazes or the contrary?
1997	Wiersma, Lagendijk	3D localization of infrared light 
> 1998	Cao, Wiersma etal, Soukoulis, Sebbah, …	Random lasering from (pre) localized states
2000	Genack Beenakker, … Van Tiggelen/ Lagendijk/Wiersma	Statistics in localized regime (exp) Idem (theo) D(r) in localized regime
2006	Maret	Anomalous dynamic transmission of light near mobility edge
2007	Fishman/Segev	Transverse Light Localization in 2D lattices
2008	Page/Skipetrov/Van Tiggelen	3D localization of ultrasound

Diffusion of Waves





Diffusion = random walk of waves

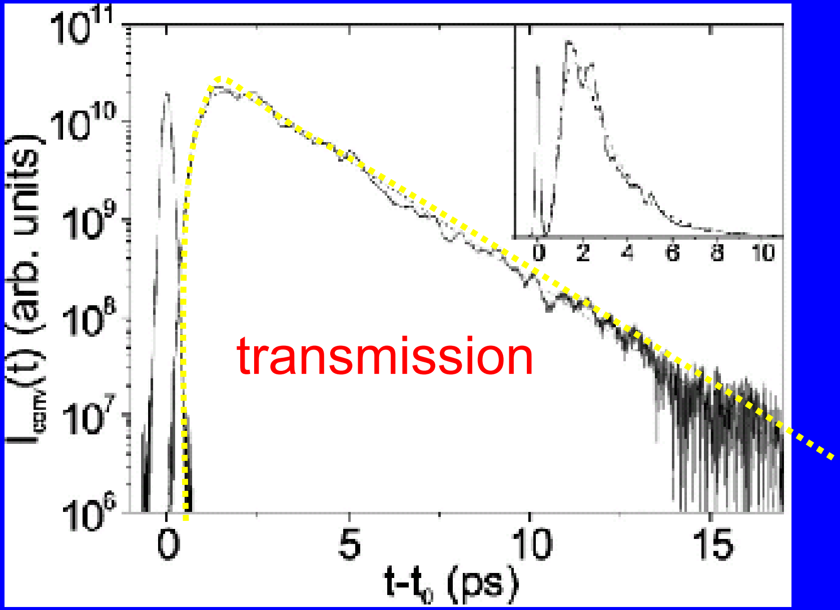
$$\boxed{\partial_t \rho(\mathbf{r},t) \; - \; D \; \nabla^2 \rho(\mathbf{r},t) \; = S \;\; \delta(t) \;\; \delta(\mathbf{r}-\mathbf{r}_S)}$$

$$\boxed{\left\langle \mathbf{r}^2(t) \right\rangle = \frac{\left\langle \rho(\mathbf{r},t)\mathbf{r}^2 \right\rangle}{\left\langle \rho(\mathbf{r},t) \right\rangle} = \; 6D\,t}$$

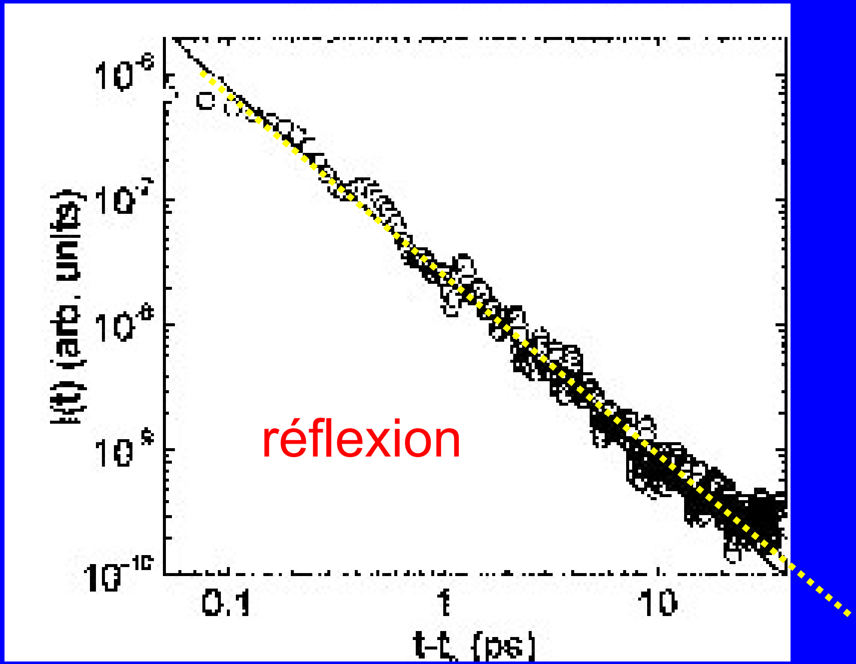
$$\boxed{D = {}^{1}\!/_{3} \;\; \mathrm{v} \;\; \ell^*}$$

diffusion constant

Diffusion, works even better than expected!



Diffusion equation



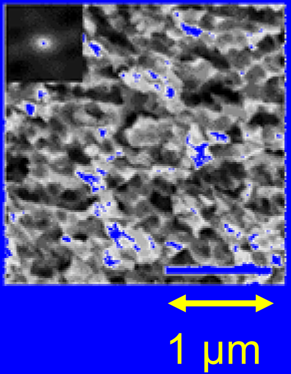
$1/t^{3/2}$

GaP poreux

$$L = 20\ \mu m$$

$$\lambda = 739\ nm$$

Lagendijk etal, PRE 2003



$$D = 23\, m^2 / s$$

$$\ell^* = 250\ nm \quad (k\ell^* = 2.1)$$

Anderson localisation?

« Unrecognizable monster »

- *suppressed diffusion : $r^2(t) \rightarrow$ constant « $D(t) \sim 1/t$ »*

Mesurable?

- *Dense « pure point » spectrum /*
  (no level repulsion: infinite media)

Mesurable?

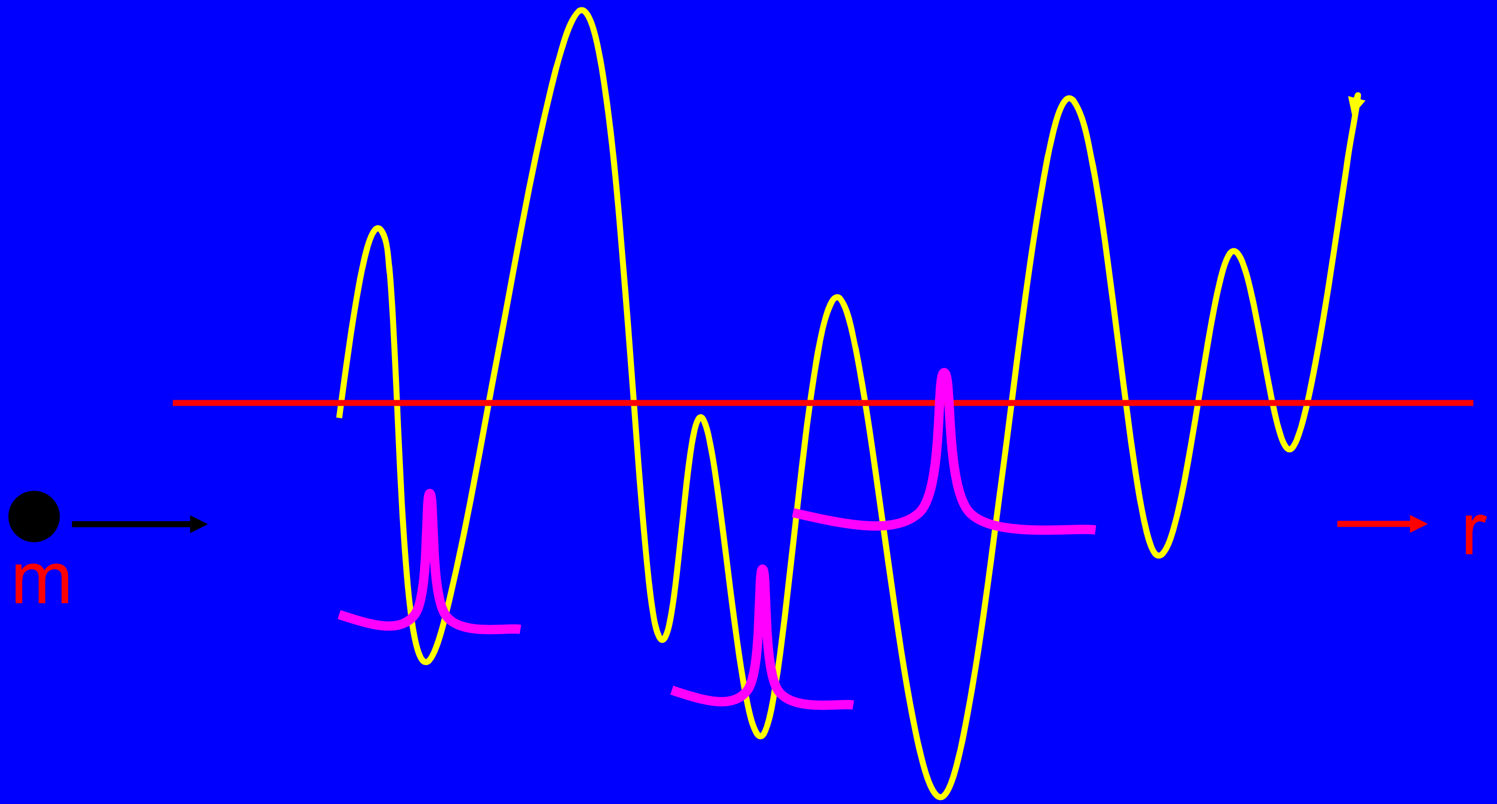
- *Giant nongaussian fluctuations (lognormal)*
- *complex dynamics (open media): $T(t) \sim \exp(-D(t)\, t / L^2)$*
  (anomalous leaking)

$R(t) \sim 1/t^2$

- *$T(L)$, $G(L) \sim \exp(-L/\xi)$*

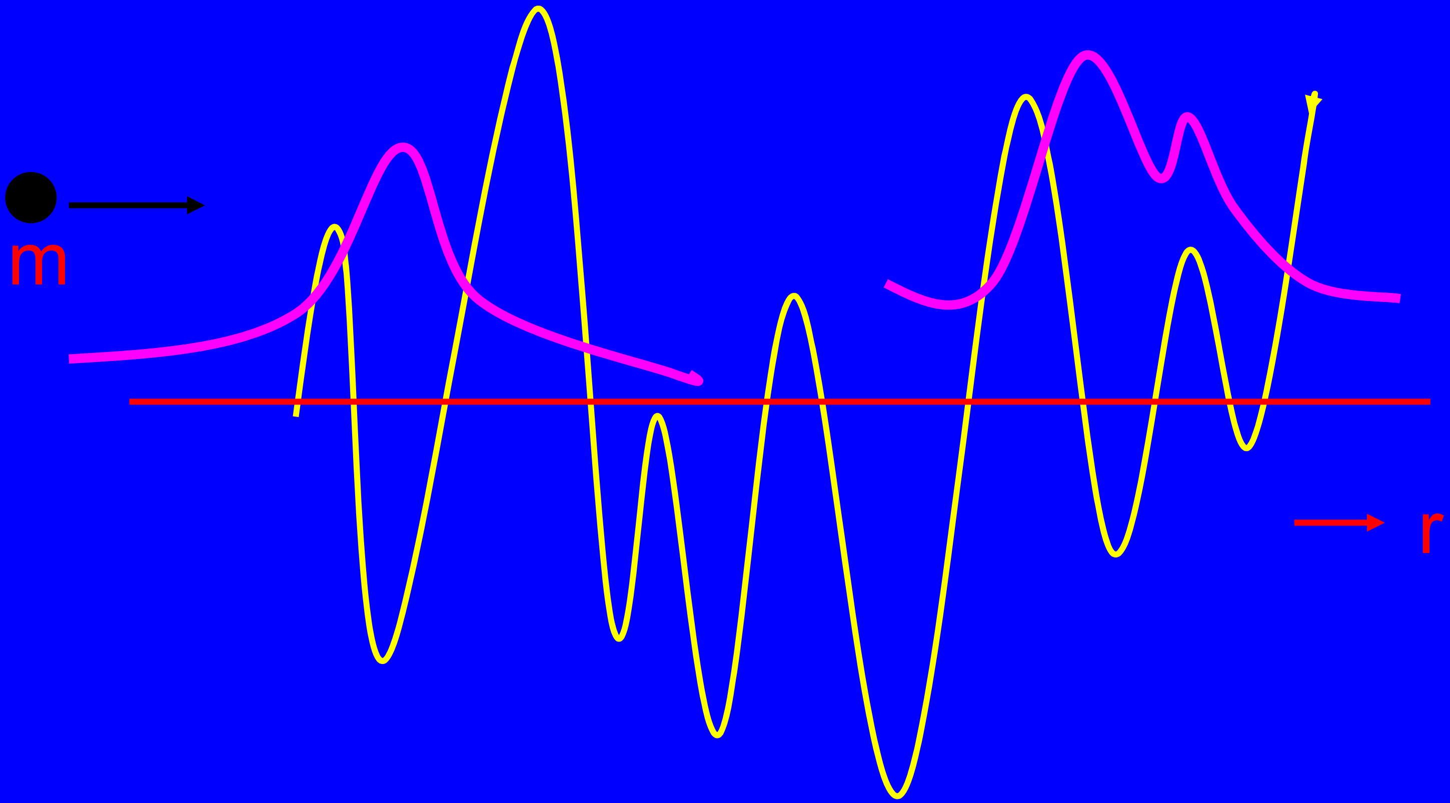
- **Need good observable with continuous control parameter: L,t, f, E ..**
- **Need a good theory that allows engineering**

1D matter wave

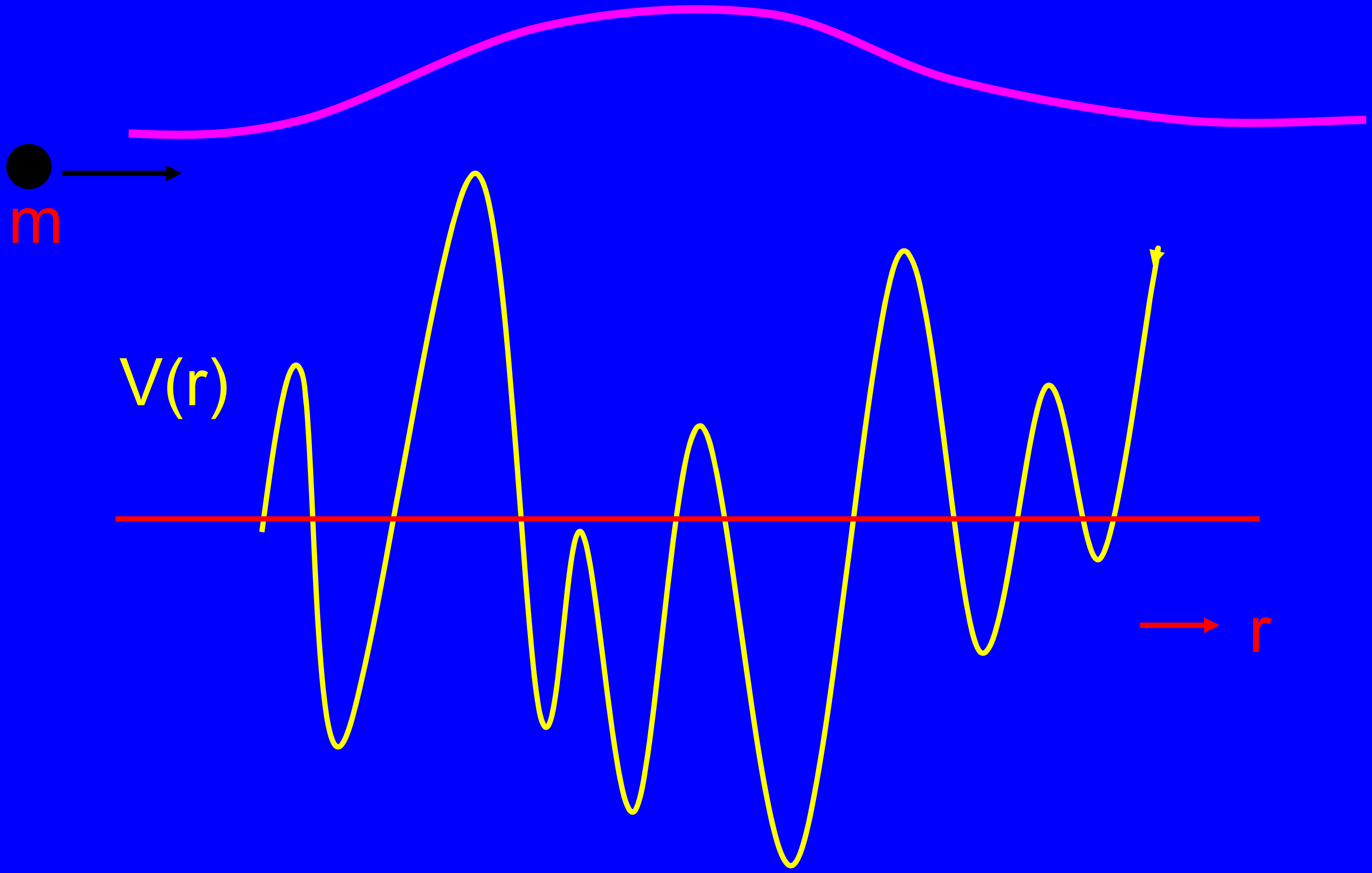


Trivial Localization: tunnelling is inefficient

1D matter wave



Localisation is «not trivial yet reasonable »



Localisation is « non trivial »: phase!

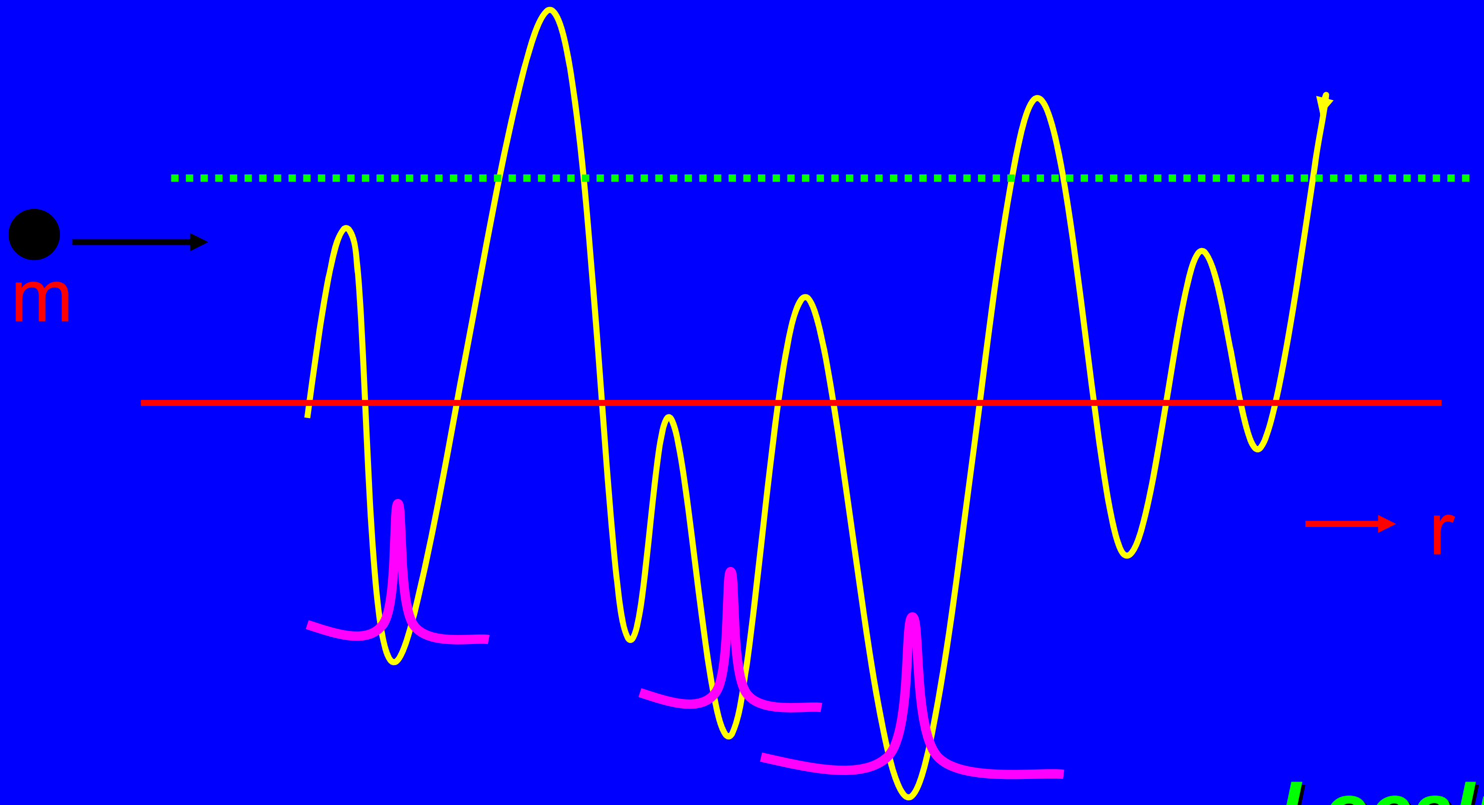
Classical waves

$$\varepsilon(\mathbf{r})\partial_t^2\psi(\mathbf{r},t)-\nabla^2\psi(\mathbf{r},t)=0$$

$$\psi=\psi(\mathbf{r})\exp(-i\omega t)\Rightarrow\begin{cases}V(\mathbf{r})=[1-\varepsilon(\mathbf{r})]\omega^2\\ E=\omega^2\end{cases}$$

$$E>V$$

3D: strong disorder



Extended states

Mobility edge $k(E)\ell(E)=1$

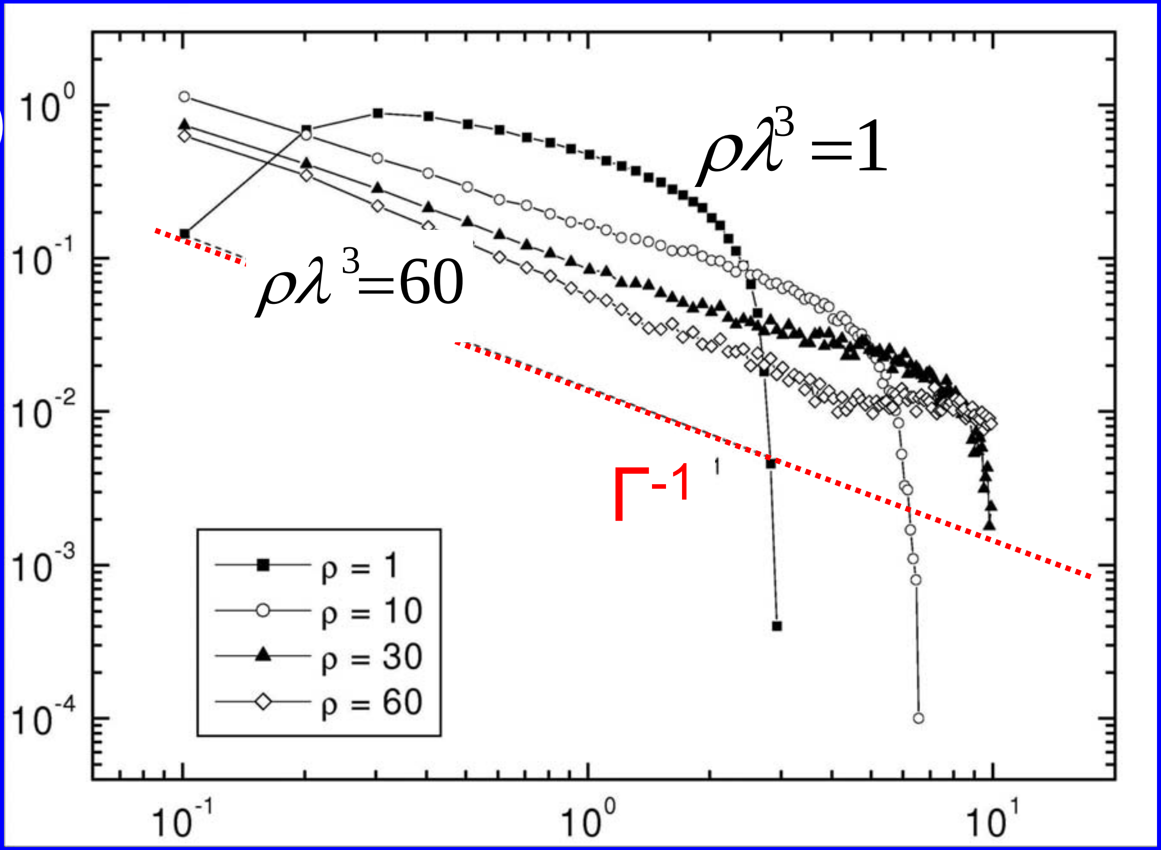
Localized states

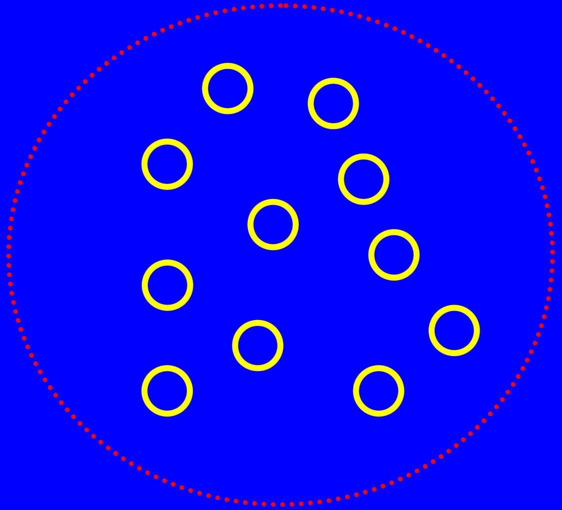
Localization in open media

Distribution of leakage rates

N dipoles in a 3D sphere
Kottos & Weiss 2003; Pinheiro/van Tiggelen PRE 2004

P(Γ)



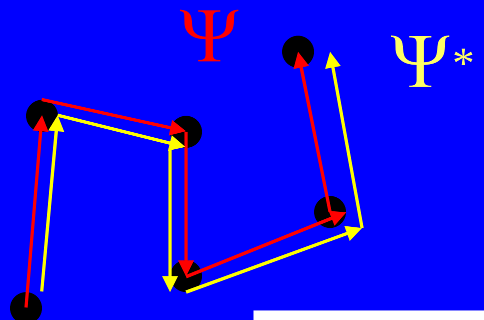


$$P(\Gamma) \propto \frac{1}{\Gamma}$$

Leakage rate/ Thouless frequency

Theoretical Description

$$k\ell \gg 1$$



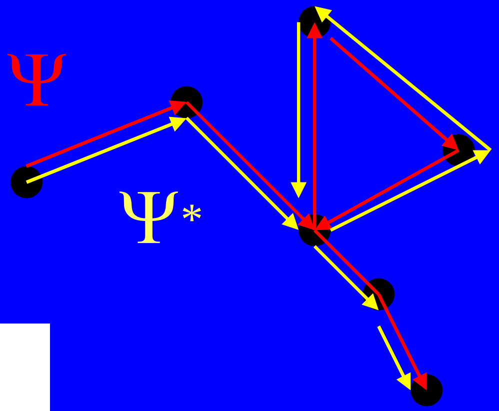
Boltzmann equation

Schwarzshild/Milne , 1900
Chandrasekhar, 1950
Van de Hulst, 1950

$$\boxed{D_B = \tfrac{1}{3} v_E \, \ell^*}$$

$$k\ell \approx 1$$

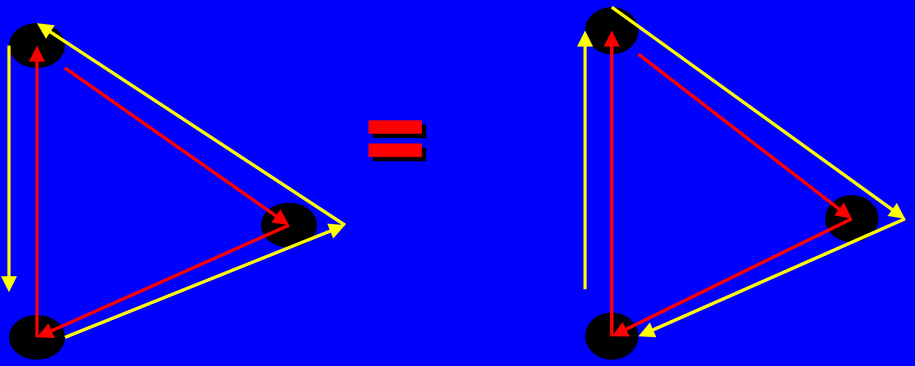
Self consistent theory



$$\boxed{\frac{1}{D} = \frac{1}{D_B} + \frac{C(\mathbf{r},\mathbf{r})}{\pi v_E N(\omega)}}$$

Vollhardt & Wölfle, 1980

reciprocity $\Rightarrow$ $C(\mathbf{r},\mathbf{r}) = G(\mathbf{r},\mathbf{r})$



$$-\nabla\cdot D(\mathbf{r})\nabla G(\mathbf{r},\mathbf{r}') = \frac{4\pi}{\ell}\delta(\mathbf{r}-\mathbf{r}')$$

$$\frac{1}{D(\mathbf{r})} = \frac{1}{D_B} + \frac{G(\mathbf{r},\mathbf{r})}{\pi v_E N(\omega)}$$

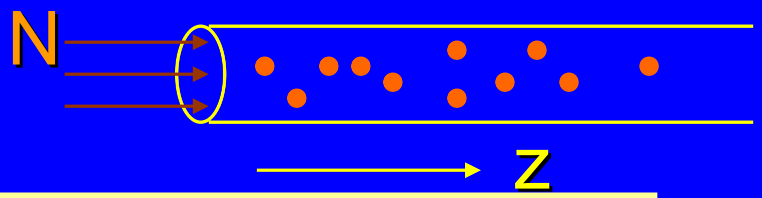
3D unbounded medium

$$G(\mathbf{r},\mathbf{r}') = G(\mathbf{r}-\mathbf{r}');\ G(\mathbf{r},\mathbf{r}) = G(0) = \frac{4\pi}{\ell}\int_{q<1/\ell} d^3\mathbf{q}\,\frac{1}{Dq^2}$$

$$\Rightarrow \boxed{D = D_B\left(1-\frac{1}{(k\ell)^2}\right)}$$

OK Mott

Quasi-1D half-wire



$$d\tau \equiv \frac{dz}{D(z)} \Rightarrow -\partial_\tau^2 G(\tau,\tau') = \frac{4\pi}{\ell}\delta(\tau-\tau') \Rightarrow G(\tau,\tau) = \frac{4\pi}{\ell}\tau$$

$$\Rightarrow \frac{d\tau}{dz} := \frac{1}{D} = \frac{1}{D_B} + \frac{2}{\xi}\tau \Rightarrow \boxed{D(z) = D_B\exp\left(-\frac{2z}{\xi}\right)}$$

$(\xi \propto N\ell)$ OK DMPK

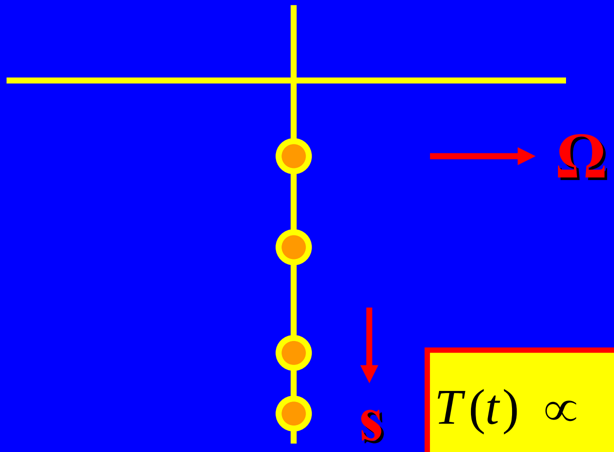
Complex Dynamics in finite open media

Skipetrov & Van Tiggelen, PRL 2004,2005

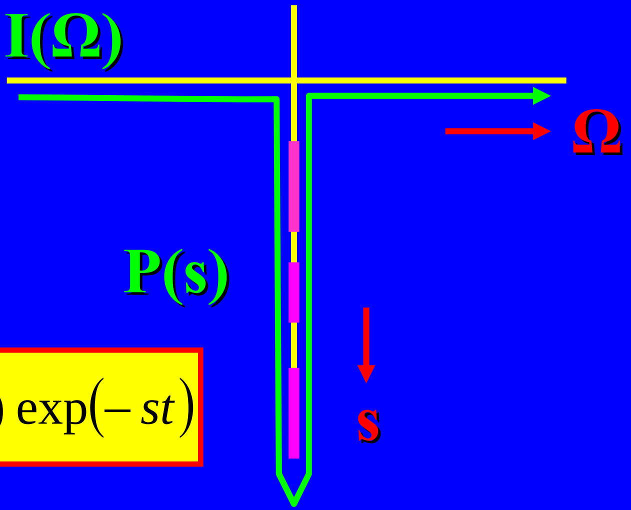
$$-s\,G(z,z',q,s) + \partial_z D(z,s)\,G(z,z',q,s) + q^2 G(z,z',q,s) = \delta(z-z')$$

$$\frac{1}{D(z,s)} = \frac{1}{D_B} + \frac{2}{k^2\ell}\int_{q</3\ell} d^2\mathbf{q}\; G(z,z,q,s)$$

Complex Frequency Ω+is



$$T(t) \propto \int_0^\infty ds\, G(z_s,z,s)\exp(-st)$$

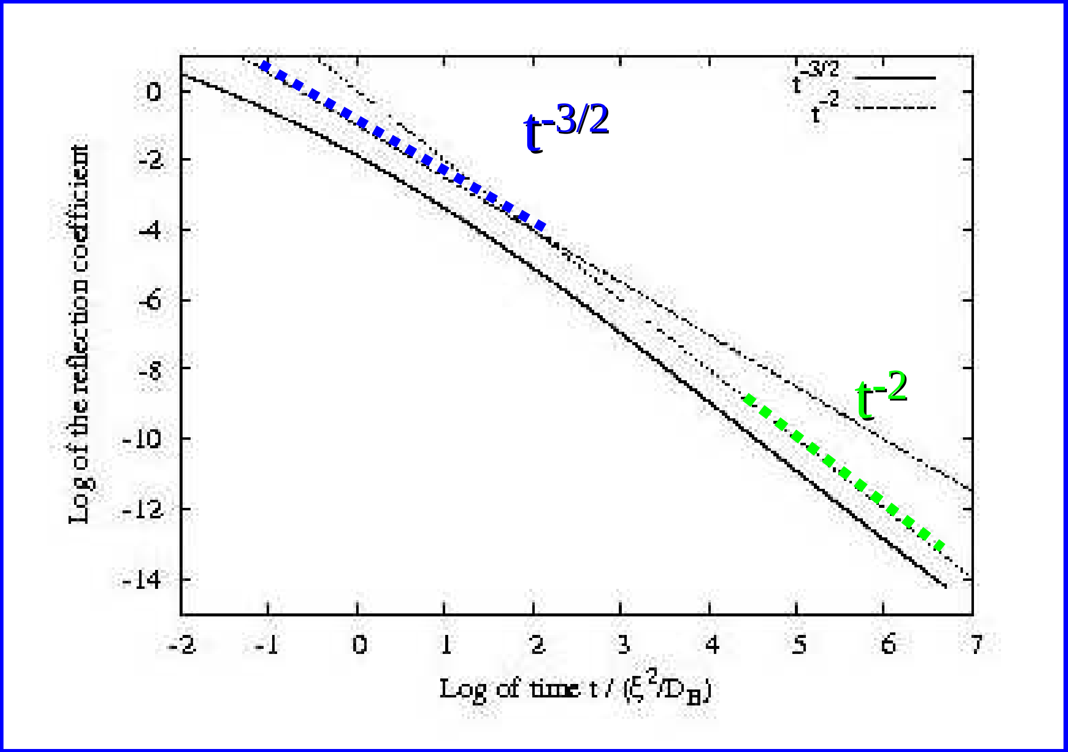


Diffuse regime: simple poles

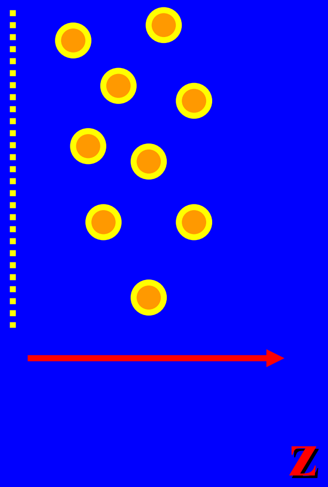
Localized regime: Riemann sheets

3D, localized Half space : kℓ=0.7

R(t) ↑



→ time/(ζ^2/D_B)

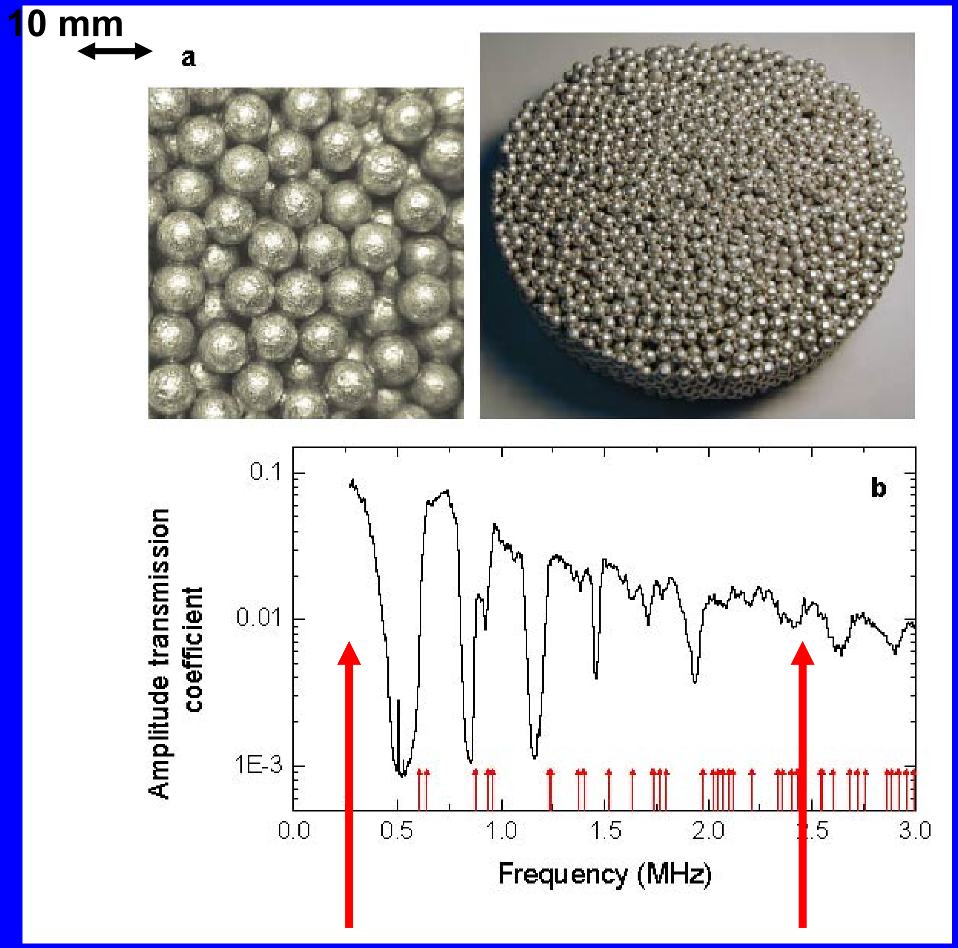


$$\boxed{R(t) \propto \frac{1}{t^2}}$$

1D sismologie : Sheng Papanicolaou, 1987
Q1D (DMKP) Titov, Beenakker, 2000

3D Transverse localisation of ultrasound

Page (Winnipeg), Skipetrov, Van Tiggelen, Cherroret



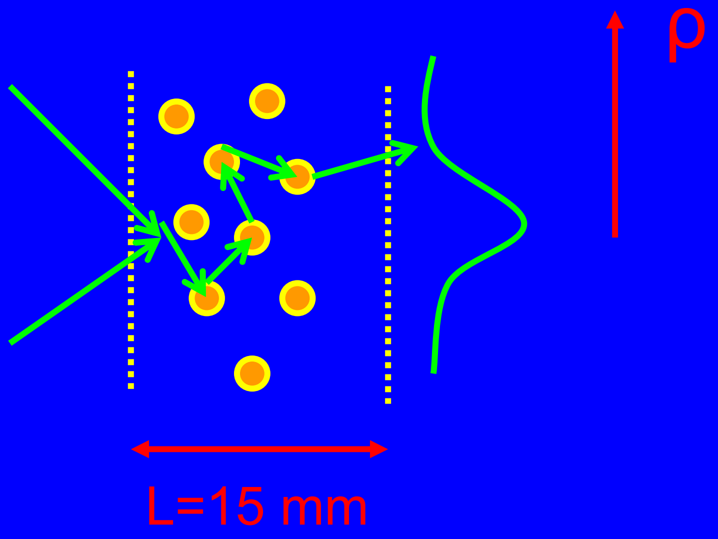
8 – 23 mm

diffuse

localized

3D Transverse localisation of ultrasound

Page (Winnipeg), Skipetrov, Van Tiggelen, Cherroret



Diffuse: $\langle \rho^2 \rangle = 4Dt$

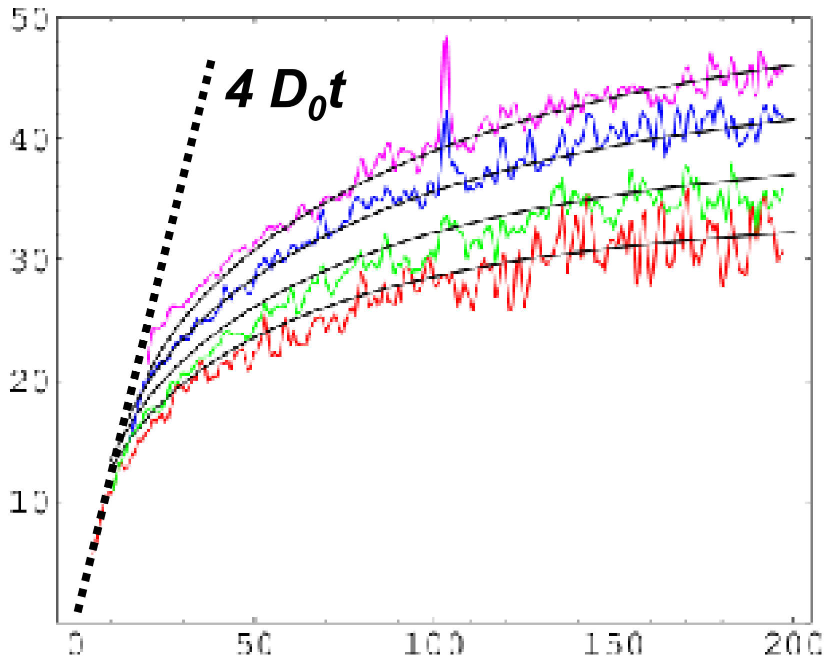
transition: $\langle \rho^2 \rangle \sim L^2$, not $t^{2/3}$

Localized: $\langle \rho^2 \rangle \sim L\xi$

3D Transverse localisation of ultrasound

Page , Skipetrov, Van Tiggelen,
Nature Physics October 2008

Transverse size



Time (ms)

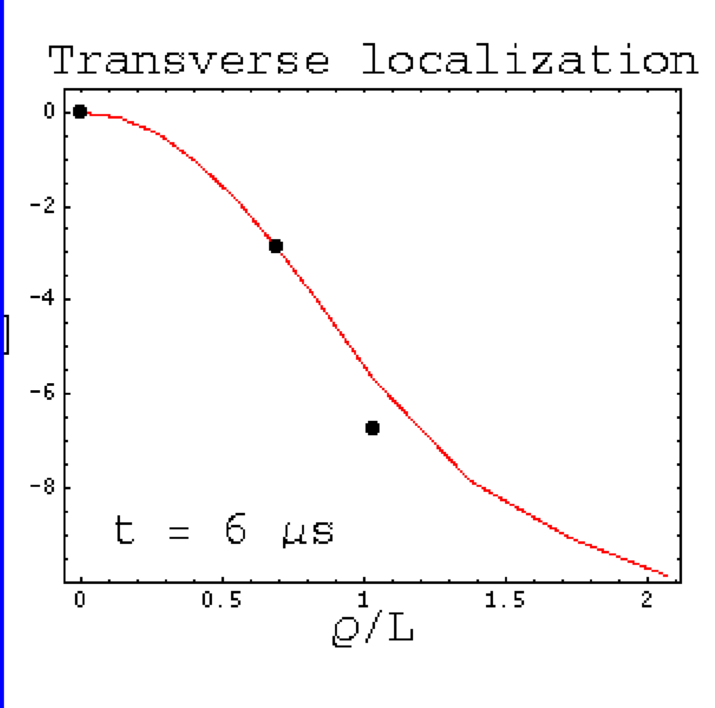
ρ = 30 mm
ρ = 25 mm
ρ = 20 mm
ρ = 15 mm

$k\ell \approx 1{,}82$

v_E=17,4 km/s=3.5 v_p

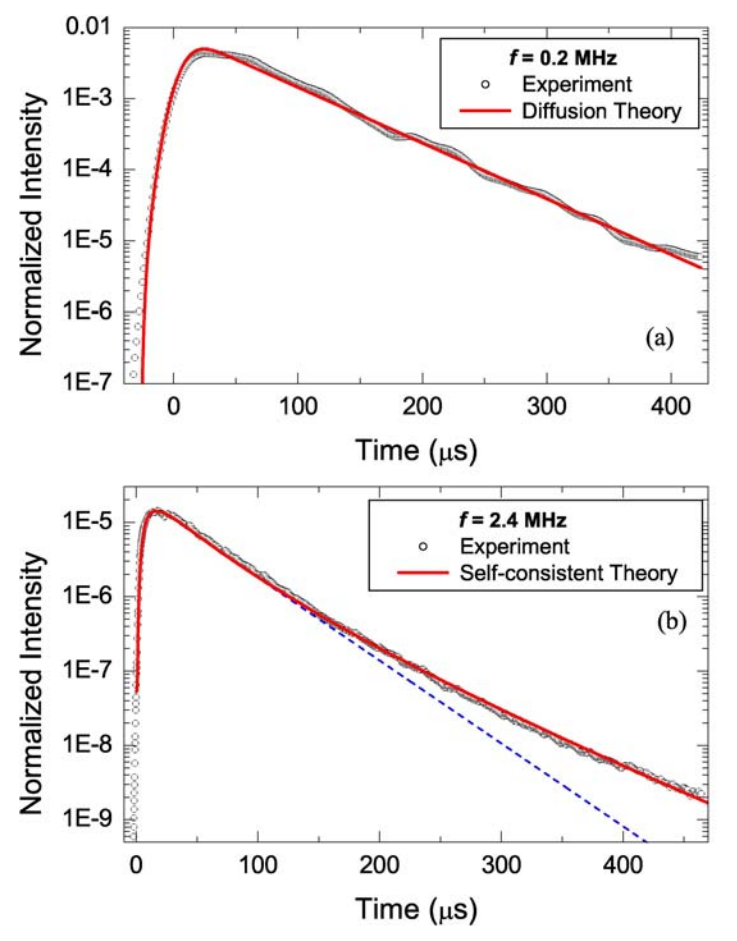
ξ= 16,3 mm

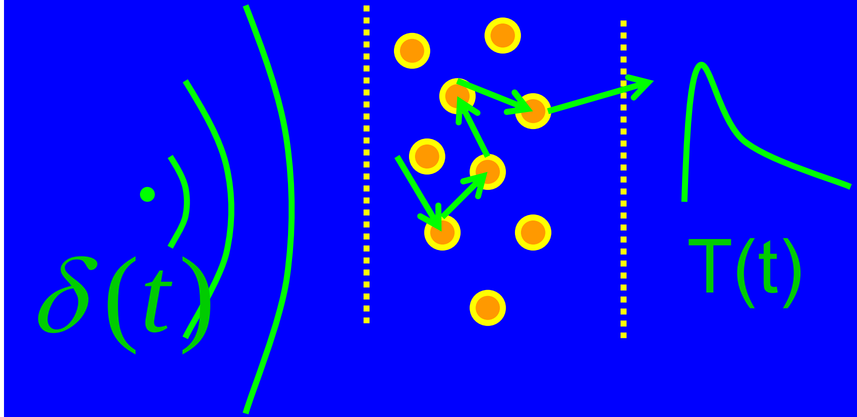
Near field
Ultrasound
energy



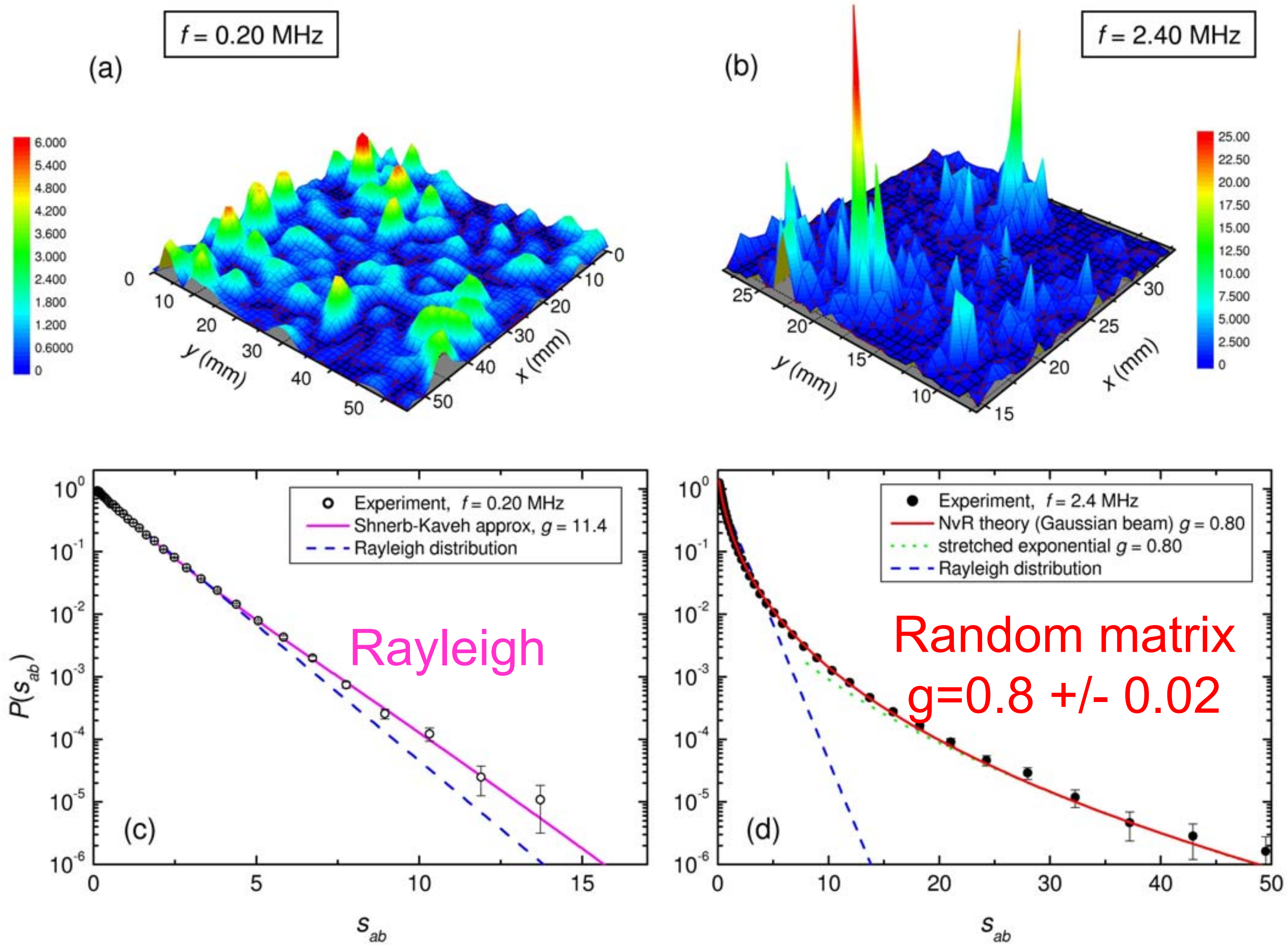
Transverse position

Time-dependent transmission





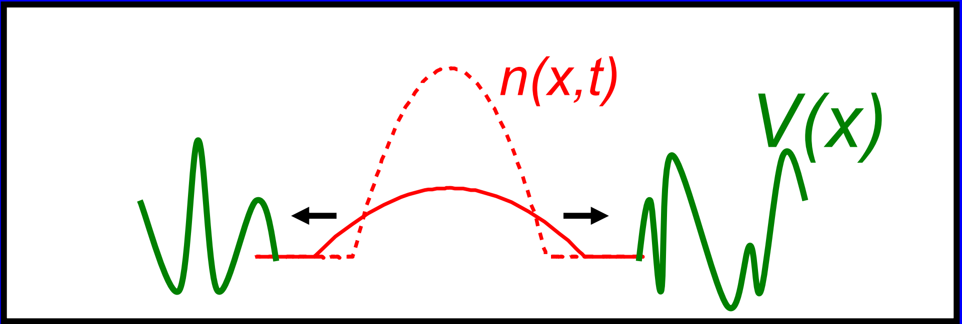
Speckle distribution of transmission



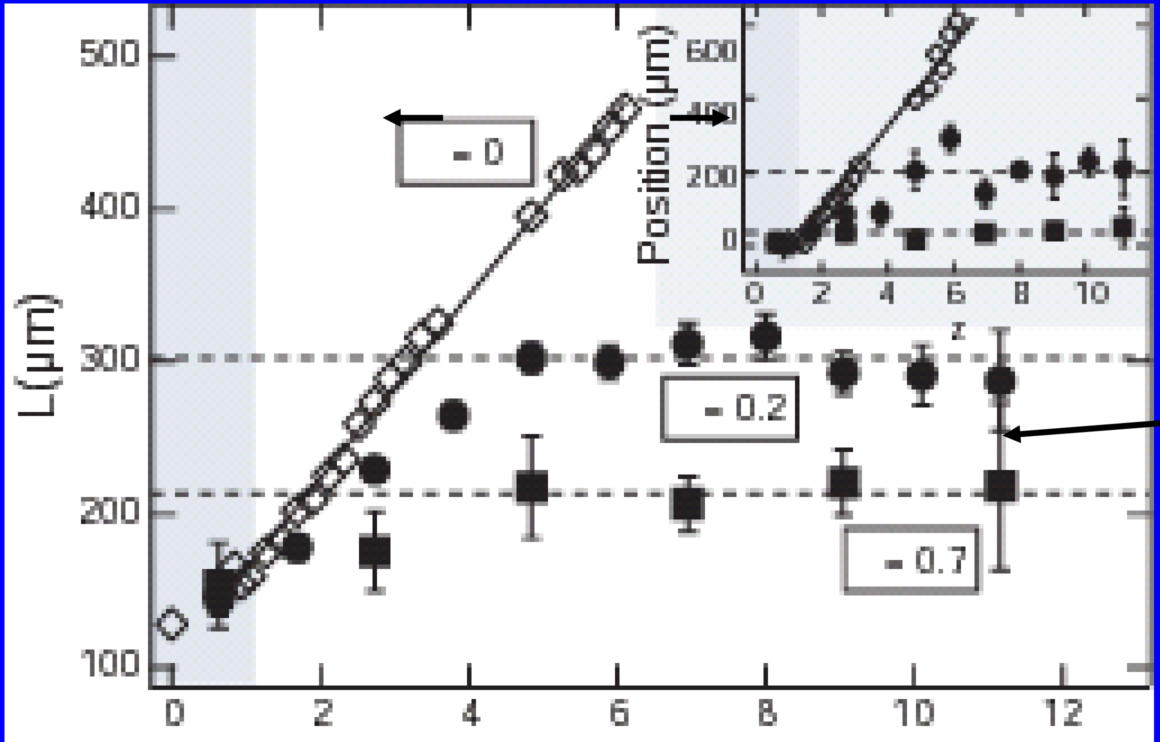
Rayleigh

Random matrix
g=0.8 +/- 0.02

Inhibition of transport of Q1D BEC in random potential



expansion



Time after trap extinction

$$\frac{\delta V}{\mu} < 1$$

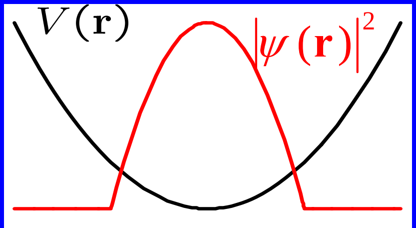
Orsay group, Firenze group
PRL oct 2005

Localization of noninteracting cold atoms in 3D white noise

with Sergey Skipetrov, Anna Minguzzi and Boris Shapiro

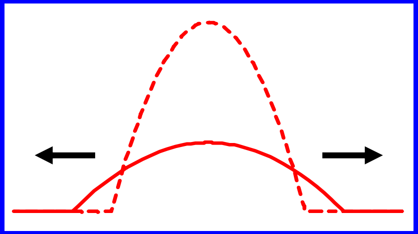
$$\boxed{i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + g\,|\psi(\mathbf{r},t)|^2\right]\psi(\mathbf{r},t)}$$

$$\boxed{\int d^3\mathbf{r}\,|\psi(\mathbf{r},t)|^2 = N}$$



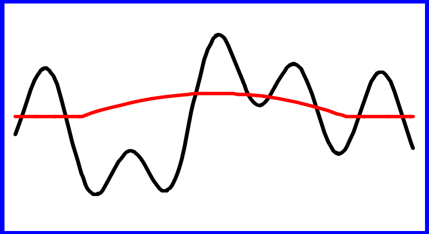
Trap stage

$$\psi(\mathbf{r},t) = e^{-i\mu t}\sqrt{\frac{\mu - V(\mathbf{r})}{g}}$$



expansion stage (t=0)

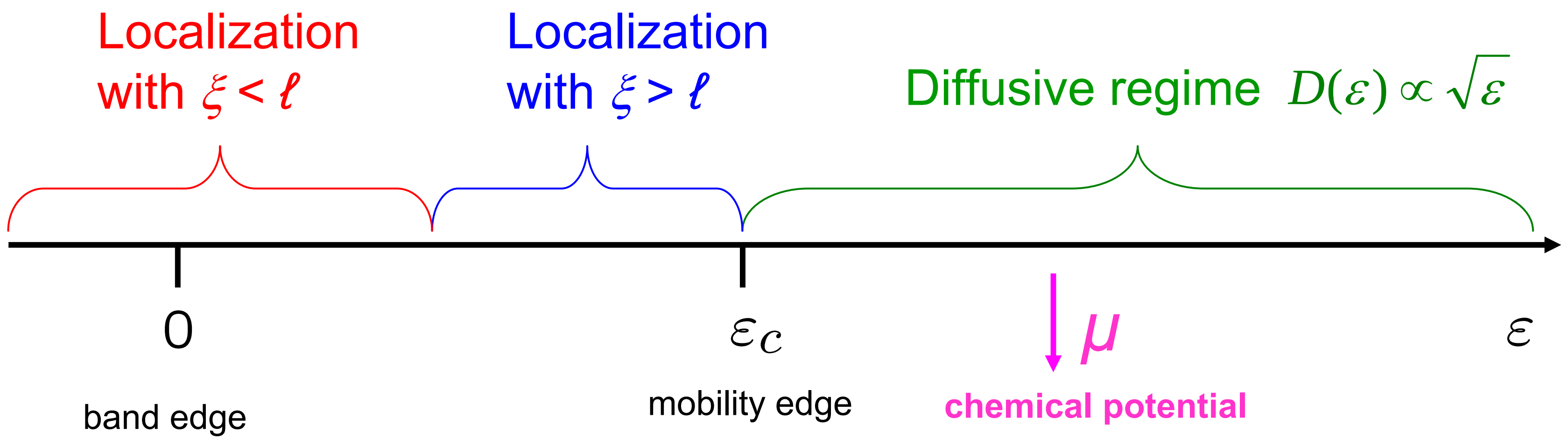
$$\psi(\mathbf{k},t=0) \propto \theta\left(\mu - \frac{\hbar^2 k^2}{2m}\right)$$



Random potential

$$n(\mathbf{r},t) = |\psi(\mathbf{r},t)|^2 = ??$$

$$t \to \infty$$



Localization with $\xi < \ell$

Localization with $\xi > \ell$

Diffusive regime $D(\varepsilon) \propto \sqrt{\varepsilon}$

0

band edge

ε_c

mobility edge

μ

chemical potential

ε

Density profile of atoms at large times

$$\left\langle V(\mathbf{r})V(\mathbf{r}')\right\rangle = 4\pi U\;\delta(\mathbf{r}-\mathbf{r}')$$

Probability of quantum diffusion

$$n(\mathbf{r},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left|\phi(\mathbf{k})\right|^2\int_{-\infty}^{\infty} dE\;A(E,\mathbf{k})\,P_E(\mathbf{r},t)$$

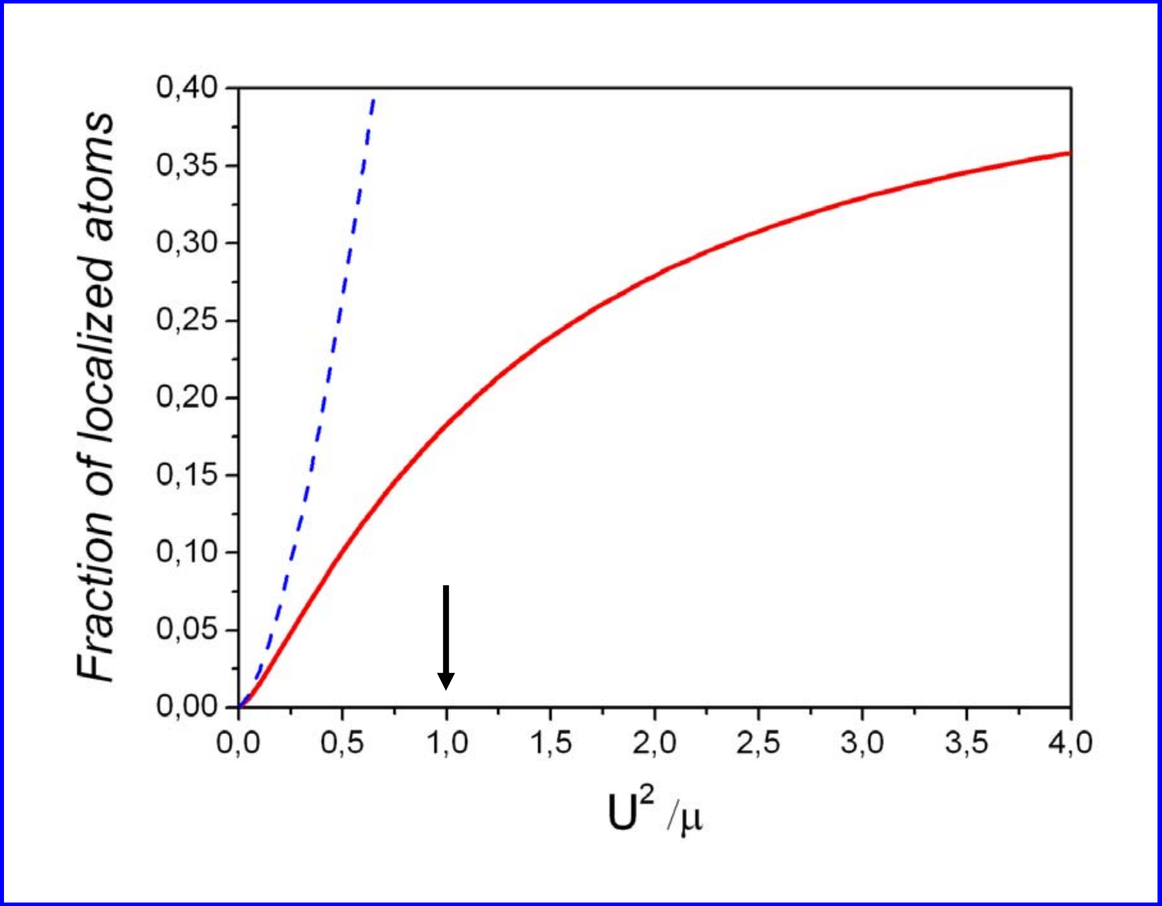
Spectral function

$$-\frac{1}{\pi}\,\mathrm{Im}\,\frac{1}{E-k^2-U^2/4+iU\sqrt{E}}$$

Distribution of velocities
mv = ħk

Mean free path: $\ell = \dfrac{1}{U}$ $\Longrightarrow$ $E < U^2$ **localized**

Density profile of atoms at large times



**Fraction
Of
Localized atoms**

Potential fluctuation/chemical potential

Density profile of atoms at large times

$$n(\mathbf{r},t) = n_{\text{loc}}(\mathbf{r}) + \Delta n_{AD}(\mathbf{r},t)$$

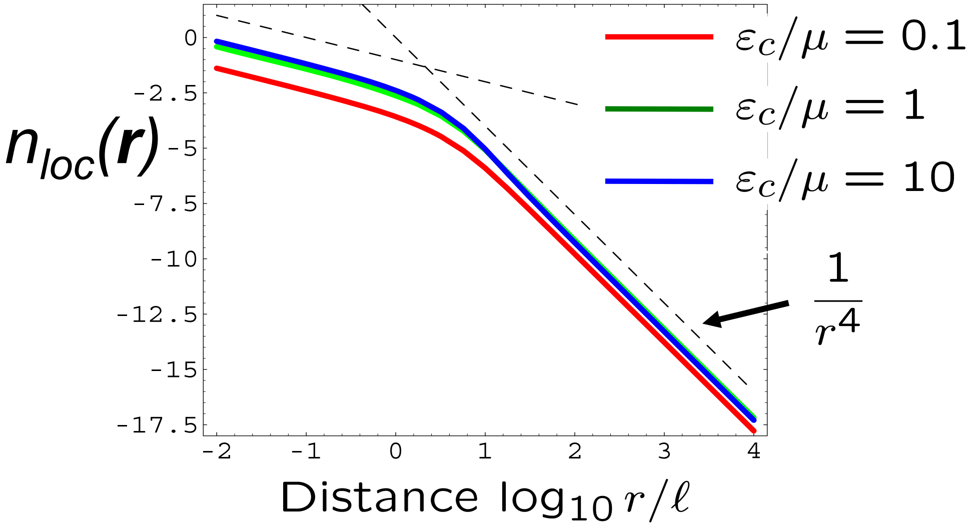
$\downarrow$ localized

$\downarrow$ anomalous diffusion

$$\boxed{\xi(\varepsilon) \propto \frac{1}{(\varepsilon_c - \varepsilon)^{\nu}} \Rightarrow n_{\text{loc}}(\mathbf{r}) \propto \frac{1}{r^{3+1/\nu}}}$$

$$\boxed{D(\varepsilon) \propto (\varepsilon - \varepsilon_c)^{s} \Rightarrow \Delta n_{AD}(\mathbf{r},t) \propto \frac{1}{r^{3-2/s}\, t^{1/s}}}$$

Selfconsistent theory (ν=s=1)

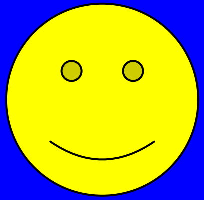


50 years of Anderson Localization
Paris, Institut Henry Poincaré
4/5 december 2008

http://www.andersonlocalization.com

Speakers: P.W.Anderson, D.J. Thouless
A. Aspect, S. John, P. Wölfle,
G. Maret, R. Weaver, A.Z. Genack,
A. Lagendijk, A. Mirlin, M. Schreiber

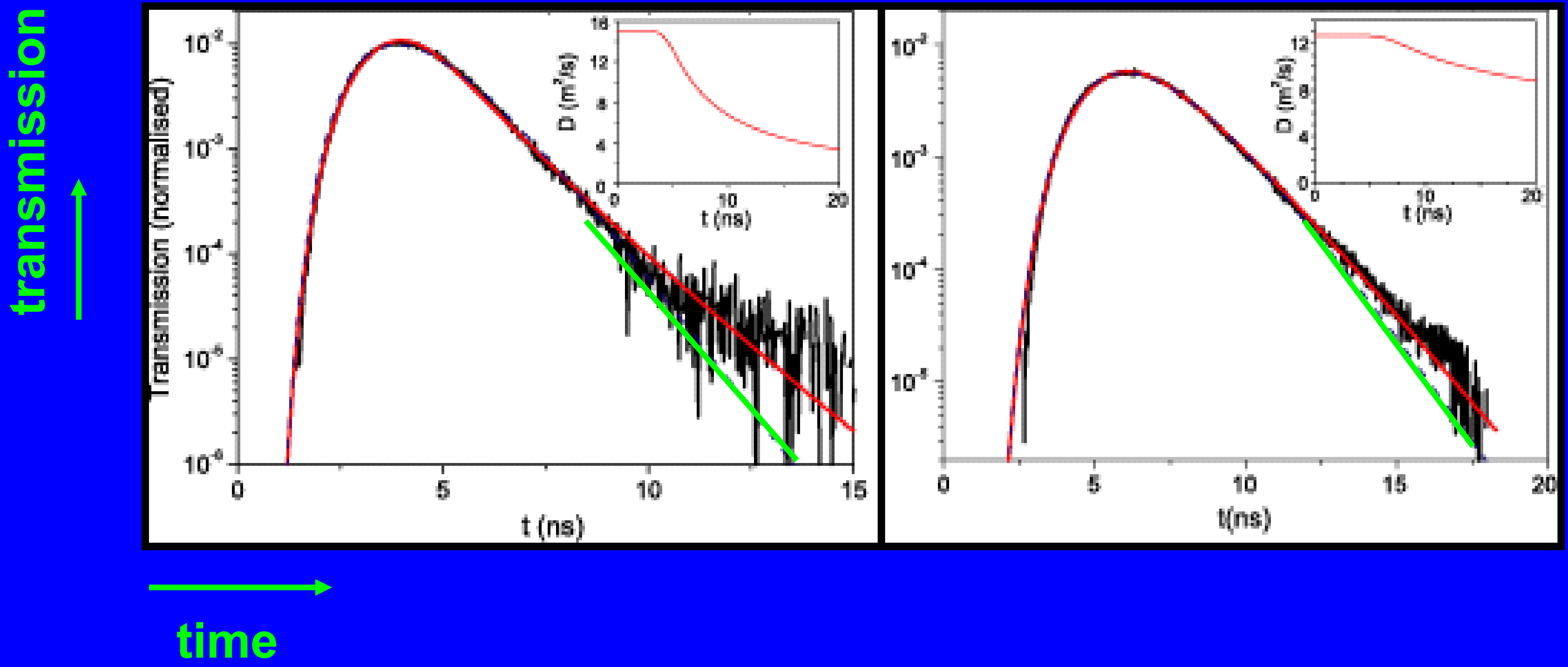
Theory and experiments, old and new things

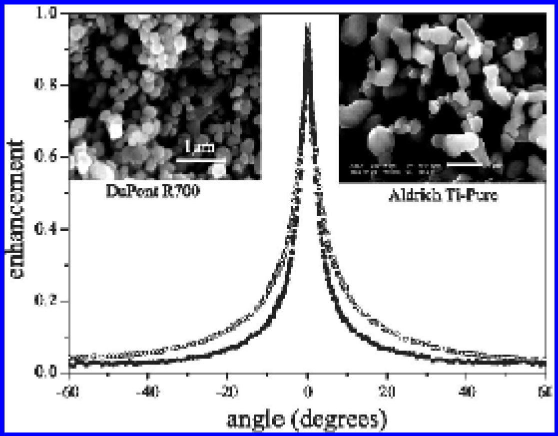


G. Maret etal, EPL 2006

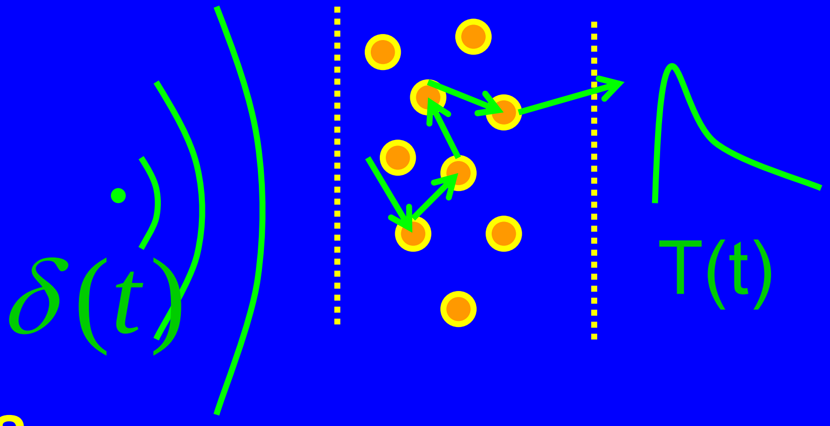
TiO$_2$ powder ; L/ℓ =10 000

Localized? : $k\ell < 4{,}2$ $D(t) \sim 1/t$ **Mobility edge: $k\ell \approx 4{,}2$ $D(t) \sim 1/t^{1/3}$**





Sample and CBS

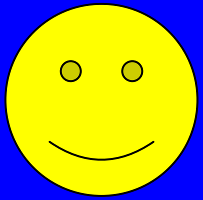


Cliché, cliché !

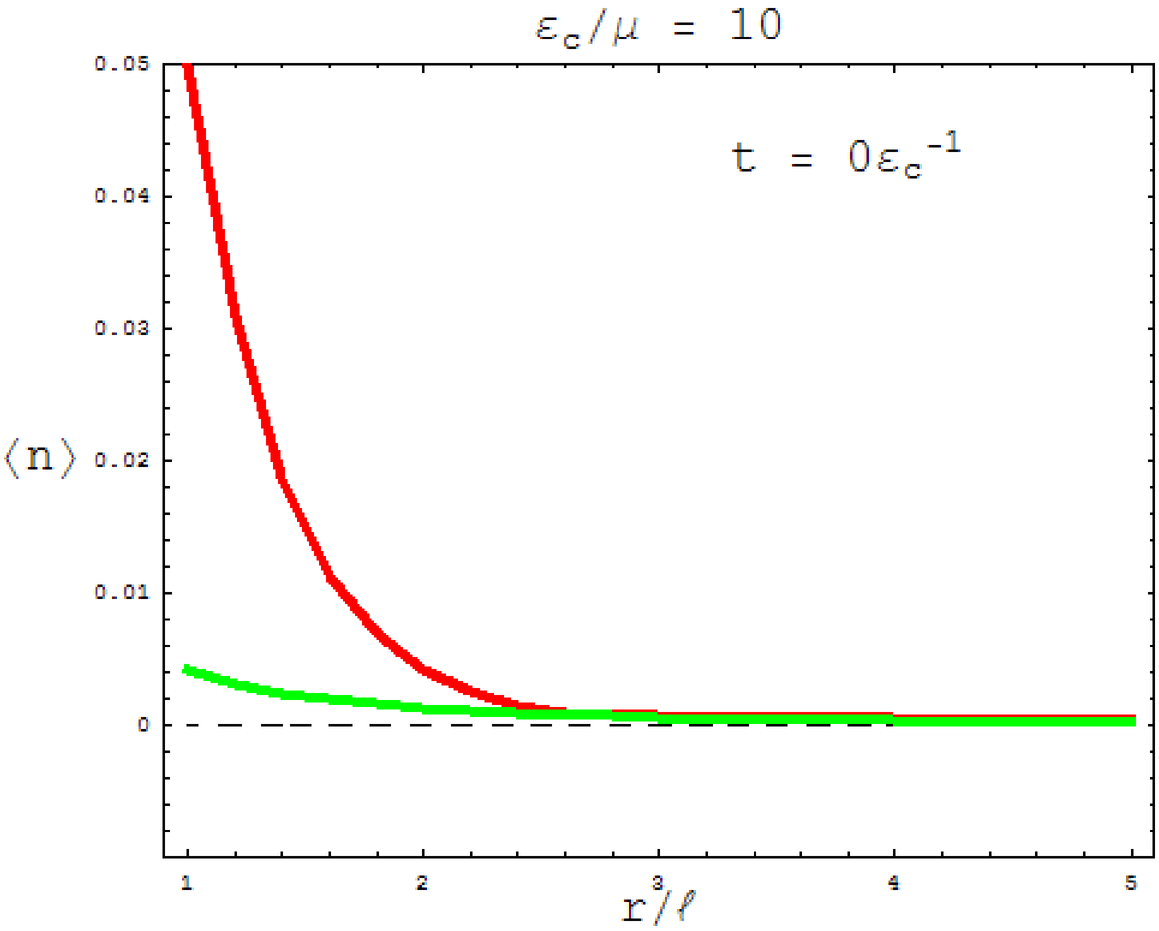
- L’article d’Anderson’s en 1958 a été « *often quoted but never read* »
- Il y a autant de définitions et d’approches qu’il y a de publications.
- La moitié des publications est …..
  - ….. soit fausse
  - ….. soit triviale
  - ….. soit déjà contenue dans le papier d’Anderson en 1958
  - ….. soit trop compliquée ou trop phénoménologique pour comprendre les expériences

Protect me from knowing what I don’t need to know
Protect me even from knowing that there are things to know that I don’t know
Protect from knowing that I decided not to know
about the things that I decided not to know about

Douglas Adams, *Mostly Harmless*, 1992



density of atoms



distance