On the Stability of Fermionic Systems with Zero-Range Interactions

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Outline

1. General setting and background [FM, DFT,FT]:
   - Quantum systems with zero-range interactions;
   - The Thomas effect;
   - Approximation by regular potentials (Efimov effect).
2. Main results [CDFMT]: $N+1$ fermions with masses $1+m$.
   - Case $N=2$: critical mass $m_\star$ for the stability.
   - Case $N>2$: stability & instability conditions.
3. Conclusions and perspectives.

Main References

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- [DFT] G. Dell'Antonio, R. Figari, A. Teta, *Ann. Inst. H. Poincaré Phys. Théor.* **60** (1994).
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Zero-Range Interactions

We want to give a *rigorous* meaning to the *formal* Schrödinger operator

$$\mathcal{H} = -\sum_{i=1}^{N} \frac{1}{2m_i} \Delta_i + \sum_{i<j} \mu_{ij} \delta(\mathbf{x}_i - \mathbf{x}_j),$$

for $\mathbf{x}_i \in \mathbb{R}^3$ and study its spectral properties.

(Physics) Motivations

- Bose-Einstein condensation and cold Bose gases [Gross '61], [Pitaevksii '61].
- Ultra-cold Fermi gases at BEC/BCS crossover: unitary limit of an approximating potential with range $\to 0$ and scattering length $\to \infty$ [Castin, Werner '06].
- Few-body Fermi systems in the unitary limit: Efimov (Thomas) effect [Braaten, Hammer '06], [Castin *et al* '10].

Symmetric vs. Self-Adjoint

A (closed) operator on a Hilbert space $\mathscr{H}$ (e.g., $=L^2(\mathbb{R}^{3N})$):

- The domain $\mathscr{D}(A^*)$ of the adjoint A^* is defined as
$$\{\phi \in \mathscr{H} \mid \forall \psi \in \mathscr{D}(A), \exists \xi \in \mathscr{H}, (A\psi, \phi) = (\psi, \xi)\},$$
and $A^* \phi = \xi$.
- A symmetric if $A = A^*$ and self-adjoint if $A = A^*$ *and* $\mathscr{D}(A) = \mathscr{D}(A^*)$.
- In general $\mathscr{D}(A) \subset \mathscr{D}(A^*)$ and if $\mathscr{D}(A)$ is enlarged then $\mathscr{D}(A^*)$ gets smaller...in *some* cases one can find a s.a. extension.
- Self-adjointness is crucial to define a dynamics (unitarity of e^{-iAt}).

Example

$K = -\Delta$ with domain $\mathscr{D}(K) = H^2_0(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$:

- $\mathscr{D}(K^*) = H^2(\mathbb{R}^3 \setminus \{0\}) \cap H^1(\mathbb{R}^3)$.
- $\mathscr{D}(K) \subsetneq \mathscr{D}(K^*)$ since functions in $\mathscr{D}(K)$ has to *vanish* around $\{0\}$ whereas functions in $\mathscr{D}(K^*)$ do not have to.

Zero-Range Interactions (mathematical definition)

- One would like to associate with $\mathcal{H}$ a self-adjoint operator on $L^2(\mathbb{R}^{3N})$.
- The δ-function is *not* a potential in the usual sense.
- One could naively think of considering $\mathcal{H}_0 = -\sum \frac{1}{2m_i}\Delta_i$ on the subspace $\{\Psi \in L^2(\mathbb{R}^{3N}) \mid \Psi|_{\mathbf{x}_i = \mathbf{x}_j} = 0\}$ but such an operator is *not* s.a. but only *symmetric* $\Longrightarrow$ look for its s.a. extensions!

Example

Consider $K = -\Delta + \mu\delta(\mathbf{x})$ on $L^2(\mathbb{R}^3)$: s.a. extensions of $-\Delta$ on $H_0^2(\mathbb{R}^3)$.

- One can classify the domains $\mathscr{D}_\alpha$ of *all* s.a. extensions K_α, $\alpha \in \mathbb{R}$:

$$\mathscr{D}_\alpha = \left\{ \Psi \in H^2(\mathbb{R}^3 \setminus \{0\}) \mid \exists q \in \mathbb{C}, \Psi \underset{|\mathbf{x}| \to 0}{\longrightarrow} \frac{q}{|\mathbf{x}|} + \alpha q + o(1) \right\},$$

$$K_\alpha \Psi \simeq -\Delta \left(\Psi - \tfrac{q}{|\mathbf{x}|} \right).$$

- The *free* Hamiltonian $-\Delta$ belongs to the family and $-\Delta = K_\infty$.

The STM Extensions

$$\mathcal{H} = -\sum_{i=1}^{N} \frac{1}{2m_i}\Delta_i + \sum_{i<j} \mu_{ij}\delta(\mathbf{x}_i - \mathbf{x}_j),$$

- In analogy with the one-body case, one looks for extensions H_α, $\alpha \in \mathbb{R}$, satisfying the *boundary conditions*
  $$\Psi = \frac{q_{ij}}{|\mathbf{x}_i - \mathbf{x}_j|} + \alpha q_{ij} + o(1), \qquad \text{as } |\mathbf{x}_i - \mathbf{x}_j| \to 0.$$
  These are the so-called Skornyakov-Ter-Martirosyan (STM) extensions.
- The STM extensions are labeled by a minimal set of parameters (in general the s.a. extensions are labeled by *operators*!) and extend in a natural way the two-body interaction.
- In general the STM extensions are *only* symmetric!
- One can find s.a. extensions of the STM extension, but in general the so-obtained operators are unbounded from below!
- This is what is usually known under the name of Thomas effect...

The Thomas Effect

- For *any* $N \geq 3$ and *any* $\alpha \in \mathbb{R}$, *any* s.a. extension of $\mathcal{H}_\alpha$ is unbounded from below, i.e., the so-called Thomas effect (TE) occurs.
- For 3 bosons [Faddeev, Minlos '62] and 3 different particles [Melnikov, Minlos '91], $\exists$ a sequence of genuine three-body bound states with energy $\to -\infty$.
- The "Thomas" states are slowly decaying states but the singularity is reached when the three particles get close together.
- The Thomas effect is independent of the sign of α although for $N = 1$ there is a (single) bound state if and only if $\alpha < 0$ (attractive interaction).
- Among all possible extensions of $\mathcal{H}$ (different from the STM ones), there are some which do *not* exhibit the TE but the associated boundary conditions are *non-local*, i.e., α is an *integral operator*.
- What is the effect of the fermionic symmetry on the TE?

Approximation by Regular Potentials

- Replace $\mu_{ij}\delta(\mathbf{x}_i - \mathbf{x}_j)$ with $V_\varepsilon(\mathbf{x}_i - \mathbf{x}_j)$ for some regular $V_\varepsilon \longrightarrow \delta$.
- If $-\Delta_i - \Delta_j + V(\mathbf{x}_i - \mathbf{x}_j)$ admits a zero-energy resonance for $i \neq j$, then it can generate an effective attractive potential for the $k-$th particle and create a sequence of bound states with energy $\longrightarrow 0$ (Efimov effect) [Simon, Klaus '79, Sigal '79].
- The TE might be the natural counterpart of the Efimov effect due to the scaling in ε [MC *et al* in progress]

Example (Albeverio *et al* '81)

Consider K_α on $L^2(\mathbb{R}^3)$ and pick $K_\varepsilon = -\Delta + V_\varepsilon(\mathbf{x})$ (V regular) with $V_\varepsilon(\mathbf{x}) = \dfrac{\lambda(\varepsilon)}{\varepsilon^2} V(\mathbf{x}/\varepsilon)$.

- If in addition $-\Delta + V$ is positive and admits a zero-energy resonance, then $K_\varepsilon \longrightarrow K_\alpha$ in *norm resolvent sense* and $\alpha \propto -\lambda'(0)$.
- $\mathrm{sgn}(\alpha)$ might be negative although K_ε is a positive operator.

Fermionic Symmetry vs. Thomas Effect

- There can be *no* zero-range interaction between fermions of the *same* species $\Longleftarrow \psi$ antisymmetric $\Longrightarrow \psi|_{\mathbf{x}_i=\mathbf{x}_j} = 0$.
- System of 2 species of fermions: N fermions of mass 1 + 1 of mass m.

Physics & Math Literature

- 2 + 1 [Efimov '72, Petrov '03]: $\exists m_\star(2) \simeq 0.0735$ s.t. the TE occurs if $m < m_\star(2)$ and the system is stable for $m > m_\star(2)$.
- 3 + 1 [Castin *et al* '10]: if $m_\star(2) < m < 0.0747$, $\exists$ genuine four-body bound states with energy $\to -\infty$.
- 2 + 1 [Shermatov '03, Minlos '10]: if $m < m_\star(2)$, *any* STM extension (restricted to $l = 1$) is not self-adjoint and *any* self-adjoint extension is unbounded from below (TE).
- N + 1 [Minlos '11]: if $N > 5$ and m sufficiently large, *any* STM extension is self-adjoint and bounded from below (no TE).

Quadratic Form Approach ($N+1$ Fermions)

- We construct the quadratic form $\mathcal{F}_\alpha$ associated with the STM extension $\mathcal{H}_\alpha$ trough a *renormalization procedure*: for any $\psi \in \mathscr{D}(\mathcal{H}_\alpha)$
$$\mathcal{F}_\alpha[\psi] = (\psi, \mathcal{H}_\alpha \psi) .$$
- If $\mathcal{F}_\alpha$ is closed then it defines a *unique* self-adjoint operator $\mathcal{H}_\alpha$.
- If $\mathcal{F}_\alpha$ is not closable (unbounded from below) then *either* $\mathcal{H}_\alpha$ is not self-adjoint *or* unbounded from below (TE).

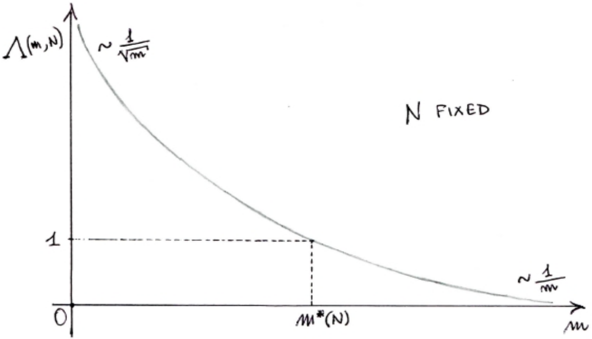
- $\mathscr{D}(\mathcal{F}_\alpha) = \left\{ \psi \in L^2_{\mathrm{f}}(\mathbb{R}^{3N}) \,\middle|\, \exists \xi \in \mathscr{D}(\Phi^\lambda_\alpha), \phi^\lambda = \psi - \mathcal{G}_\lambda \xi \in H^1_{\mathrm{f}}(\mathbb{R}^{3N}) \right\}$,
$$\mathcal{F}_\alpha[\psi] := \mathcal{F}_0[\phi^\lambda] + \lambda \|\phi^\lambda\|^2_{L^2(\mathbb{R}^{3N})} - \lambda \left\|\psi\right\|^2_{L^2(\mathbb{R}^{3N})} + N \Phi^\lambda_\alpha \left[\xi\right],$$
- $\mathcal{F}_0[\phi] = (\phi, \mathcal{H}_0 \phi)$ (in the center of mass ref. frame) and, for any $\lambda > 0$, $\mathcal{G}_\lambda \xi = (\mathcal{H}_0 + \lambda)^{-1} * \xi \delta(\mathrm{x}_i - \mathrm{x}_j)$ (note that $\mathcal{G}_\lambda \xi \notin H^1(\mathbb{R}^{3N})$).
- $\Phi^\lambda_\alpha[\xi]$ acts on the "charge" ξ ($\hat{\xi} \in H^{1/2}(\mathbb{R}^{3(N-1)})$).

Mass Threshold

Definition (Critical Mass $m_\star(N)$)

We define $m_\star(N)$ as the *unique* solution of $\Lambda(m, N) = 1$ with

$$\Lambda(m, N) = \frac{2}{\pi}(N-1)(m+1)^2 \left[\frac{1}{\sqrt{m(m+2)}} - \arcsin\left(\frac{1}{m+1}\right) \right].$$



Stability

Theorem (MC *et al* '12)

For any $N \geq 2$ and $m > m_\star(N)$, the quadratic form $\mathcal{F}_\alpha$ is closed and bounded from below. Moreover, if $\alpha \geq 0$, $\mathcal{F}_\alpha \geq 0$, whereas, if $\alpha < 0$,

$$\mathcal{F}_\alpha[\psi] \geq -\frac{\alpha^2}{4\pi^4 \left(1 - \Lambda(m, N)\right)^2} \|\psi\|_{L^2}^2.$$

Corollary

For any $N \geq 2$ and $m > m_\star(N)$, any STM extension $\mathcal{H}_\alpha$ is self-adjoint and bounded from below.

Remarks

- The infimum of $\mathcal{F}_\alpha$ is reached on charges in the susbspace $l = 1$.
- The lower bound is expected to be essentially *optimal* for $N = 2$.

Instability

Theorem (MC *et al* '12)

For any $N \geq 2$ and $m < m_\star(2)$, the quadratic form $\mathcal{F}_\alpha$ is unbounded from below for any $\alpha \in \mathbb{R}$ (Thomas effect).

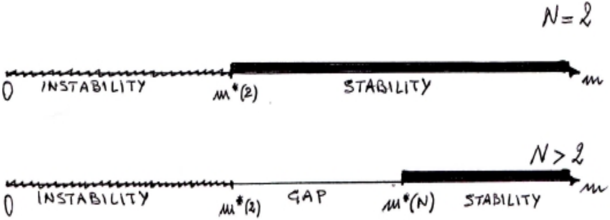
Corollary

For any $N \geq 2$ and $m < m_\star(2)$, any STM extension $\mathcal{H}_\alpha$ can not be self-adjoint and bounded from below.

Remarks

- Both cases $\mathcal{H}_\alpha$ *non* self-adjoint or $\mathcal{H}_\alpha$ self-adjoint but *unbounded from below* are a priori possible.
- The case $N > 2$ is dealt with as $N = 2$ with $N - 2$ fermions "far away".

$N+1$ vs. $2+1$



- For $N > 2$ the results are only *partial*: unknown behavior for $m_\star(2) < m < m_\star(N)$.
- One expects a different stability/instability threshold $\tilde{m}_\star(N) > m_\star(2)$ due to the occurence of N-body bound states.
- The role of the antisymmetry must be *subtle* but *crucial*.

$2+1$

- $m_\star(2)$ is the *sharp* mass threshold for the Thomas effect.
- For $\alpha < 0$, a lower bound of the form

$$\mathcal{F}_\alpha[\psi] \geq -\frac{C\alpha^2}{(1-\Lambda(m,2))}\|\psi\|^2_{L^2}$$

  is expected to be optimal [MC *et al* in progress], i.e., $\exists$ a suitable sequence of charges ξ_n such that $\mathcal{G}_\lambda \xi_n$ *saturates* the bound.
- The continuous spectrum $\sigma_{\mathrm{ac}}(\mathcal{H}_\alpha)$ is $(-(2\pi\alpha)^2, \infty) \Longleftarrow$ the bottom of $\sigma_{\mathrm{ac}}(\mathcal{H}_\alpha)$ is given by 2 particles in the bound state $+$ 1 particle far away.
- If m is close enough to $m_\star(2)$ the saturation of the bound implies the existence of at least *genuine* three-body bound state of the STM extension with energy *below* the continuous spectrum threshold.
- As $m \to m_\star(2)$ there is at least one bound state with energy $\to -\infty$.

$2+1$: Sketch of the Proof

- Positivity (coerciveness) of the charge form Φ_α^λ for some $\lambda > 0$ implies closedness of the complete form $\mathcal{F}_\alpha$.
- One has the decomposition $\Phi_\alpha^\lambda[\xi] = \Phi_{\alpha,\lambda}^{\mathrm{diag}}[\xi] + \Phi_\lambda^{\mathrm{off}}[\xi]$ with

$$\Phi_{\alpha,\lambda}^{\mathrm{diag}}[\xi] = \int \mathrm{d}\mathbf{p} \left[\alpha + 2\pi^2 \sqrt{\tfrac{m(m+2)}{(m+1)^2}\mathbf{p}^2 + \lambda}\right] \left|\hat{\xi}(\mathbf{p})\right|^2,$$

$$\Phi_\lambda^{\mathrm{off}}[\xi] = \int \mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t} \frac{\hat{\xi}^*(\mathbf{s})\hat{\xi}(\mathbf{t})}{\mathbf{p}^2 + \mathbf{q}^2 + \frac{2}{m+1}\mathbf{p}\cdot\mathbf{q} + \lambda}.$$

- Decomposition in spherical harmonics $Y_l^m(\vartheta,\varphi)$ of Φ_α^λ + diagonalization of $\Phi_\lambda^{\mathrm{off}}$ by Mellin transform $\Longrightarrow$ $\Phi_\lambda^{\mathrm{off}}$ can be negative only for l odd and the *worst* case is $l = 1$.
- $\Phi_\lambda^{\mathrm{off}}[\xi] \geq -\Lambda(m,2)\Phi_{\alpha,\lambda}^{\mathrm{diag}}[\xi] \Longrightarrow$ lower bound

$$\Phi_\alpha^\lambda[\xi] \geq (1 - \Lambda(m,2))\, \Phi_{\alpha,\lambda}^{\mathrm{diag}}[\xi]$$

which yields the condition $m > m_\star(2)$ and the lower bound on $\mathcal{F}_\alpha$:

$$\mathcal{F}_\alpha[\psi] \geq -\lambda\|\psi\|^2 + 2\left[2\pi\,(1-\Lambda)\sqrt{\lambda} + \alpha\right]\|\xi\|^2.$$

Perspectives

$2 + 1$

- ✔ Ground state energy for $m > m_\star(2)$ [MC *et al* in progress].
- ✗ Estimate of the number of bound states as $m \to m_\star(2)$.
- ✔ Approximation by regular potentials [Dell'Antonio, Michelangeli in preparation].
- ✗ Thomas effect vs. Efimov effect.

$N + 1$

- ✗ Threshold shift for $N > 2$ (role of the antisymmetry).
- ✗ Behavior for large N and effective model for $N \to \infty$.

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