3-Dimensional Anderson Localization of Elastic Waves

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Localization [...] very few believed it at the time, and even fewer saw its importance, among those who failed was certainly its author. It has yet to receive adequate mathematical treatment, and one has to resort to the indignity of numerical simulations to settle even the simplest questions about it.

P.W. Anderson, Nobel lecture, 1977

.....and now we have experiments!
My precious collaborators

- John Page, Anatoli Strybulevych, Kurt Hildebrand (Winnipeg-Canada)
- Sergey Skipetrov and Nicolas Cherroret (Grenoble-France)
- Ad Lagendijk and Sanli Faez (Amsterdam-the Netherlands)
- Diederik Wiersma (Florence-Italy)
**Diffusion of Waves**

**Diffusion = random walk of waves**

\[
\partial_t \rho(r, t) - D \nabla^2 \rho(r, t) = S \delta(t) \delta(r - r_S)
\]

\[
\langle r^2(t) \rangle = \frac{\langle \rho(r, t)r^2 \rangle}{\langle \rho(r, t) \rangle} = 6D t
\]

\[
D = \frac{1}{3} v \ell^* 
\]

diffusion constant
CONCLUDING REMARKS

\[ D(L) \]
\[ D(R) \]
\[ D(q) \]
\[ D(R) \]

THANK YOU
Diffusion, works even better than expected!

porous GaP

\[ L = 20 \, \mu m \]
\[ \lambda = 739 \, nm \]

Lagendijk et al., PRE 2003

Diffusion equation

\[ D = 23 \, m^2 / s \]
\[ \ell^* = 250 \, nm \quad (k\ell^* = 2.1) \]
Anderson localisation?

« Unrecognizable monster »

- Vanishing of diffusion: \( r^2(t) \rightarrow \text{constant} \) « \( D(t) \sim 1/t \) »

- Scale-dependence of conductance \( T(L), G(L) \sim \exp(-L/\xi) \) « \( D(L) \) »

- Anomalous, dynamic leaking \( T(t) \sim \exp(-D(t,L) t /L^2) \)
  \( R(t) \sim 1/t^2 \)

- Dense « pure point » spectrum / no level repulsion

- Giant nongaussian fluctuations

- Multifractality of wave function
Theoretical Description

\[ k \ell \gg 1 \]

**Diffusion equation**

Schwarzschild/Milne, 1900
Chandrasekhar, 1950
Van de Hulst, 1950

**Self consistent theory**

\[ k \ell \approx 1 \]

\[ \frac{1}{D} = \frac{1}{D_B} + \frac{C(r,r)}{\pi \nu E N(\omega)} \]

\[ D_B = \frac{1}{3} \nu_E \ell^* \]

Vollhardt & Wölfle, 1980
reciprocity \Rightarrow C(r,r) = G(r,r)

\nabla \cdot D(r) \nabla G(r,r') = \frac{4\pi}{\ell} \delta(r - r')

\frac{1}{D(r)} = \frac{1}{D_B} + \frac{G(r,r)}{\pi v_E N(\omega)}
3D unbounded medium

\[ G(r,r') = G(r-r'); \quad G(r,r) = G(0) = \frac{4\pi}{\ell} \int_{q<1/\ell} d^3q \frac{1}{Dq^2} \]

\[ \Rightarrow D = D_B \left( 1 - \frac{1}{(k\ell)^2} \right) \]

Quasi-1D wave guide

\[ d\tau \equiv \frac{dz}{D(z)} \Rightarrow -\partial_1 G(\tau,\tau') = 4\pi \frac{\delta(\tau-\tau)}{\ell} \Rightarrow G(\tau,\tau) = \frac{4\pi}{\ell} \tau \]

\[ \Rightarrow \frac{d\tau}{dz} := \frac{1}{D} = \frac{1}{D_B} + \frac{2}{\xi} \tau \Rightarrow D(z) = D_B \exp \left( \frac{2z}{\xi} \right) \]

\[ (\xi \propto N\ell) \]
Half space 3D
$k\ell=1$

Van Tiggelen, Lagendijk, Wiersma,
PRL 2000

\[ D(z) = \frac{D(0)}{1+z/\zeta} \Rightarrow T \propto \frac{1}{L^2} \]
Complex Dynamics in finite open media

Skipetrov & Van Tiggelen, PRL 2004, 2006

\[-s G(z, z', q, s) + \partial_z D(z, s) G(z, z', q, s) + q^2 G(z, z', q, s) = \delta(z - z')\]

\[
\frac{1}{D(z, s)} = \frac{1}{D_B} + \frac{2}{k^2 \ell} \int \frac{d^2q}{q^2 < k^2 \ell} G(z, z, q, s)
\]

Complex Frequency $\Omega + is$

Diffuse regime: simple poles

Localized regime: Riemann sheets

$T(t) \propto \int_0^\infty ds \, G(z_s, z, s) \exp(-st)$
3D, localized half space : $k\ell=0.7$

$R(t) \propto \frac{1}{t^2}$

1D sismologie: Sheng Papanicolaou, 1987
Q1D (DMKP) Titov, Beenakker, 2000
3D localisation of ultrasound
Hu, Page and Strybulevych, Skipetrov, Van Tiggelen
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Diffuse $\lambda_p=9$ mm, $\ell=2$ mm
Localized $\lambda_p=2$ mm, $\ell=0.6$ mm
3D transverse locQUAlisation of ultrasound

Transverse locCalisation: De Vries, Lagendijk and De Raedt, PRL 1989

Diffuse: $<\rho^2> = 4Dt$
Near transition: $<\rho^2> \sim L^2$, not $t^{2/3}$
Localized: $<\rho^2> \sim L\xi$, not $\xi^2$

Skipetrov and Cherroret
$T(\rho, t) = T(0, t) \exp\left(-\frac{\rho^2}{w(\rho, t)^2}\right)$

$D(t)$

$D(L)$

$T(\rho, t) / T(0, t)$

$k\ell \approx 1.82$

$v_L = 17.4 \text{ km/s}$

$= 3.5 v_p$

$\xi = 16.3 \text{ mm}$

Transverse Size $w(\rho, t)$
Speckle distribution of transmission

\[ P(I) \]

Rayleigh

\[ \frac{I}{\langle I \rangle} \]

Random matrix
Rossum & Nieuwenhuizen
\[ g = 0.8 \pm 0.02 \]
Multifractality of wave function

Faez, Page, Lagendijk and Van Tiggelen

Box counting

\[
I_b = \frac{\int_{b^d} d^d r \, I(r)}{\int_{L^d} d^d r \, I(r)} \quad \lambda \ll b \ll L
\]

\[
P_q = \sum_b (I_b)^q = \left(\frac{L}{b}\right)^{-d(q-1) + \Delta(q)}
\]

Generalized Inverse Participation Ratio

\[
\Delta(q) = \gamma q (1 - q) \quad \Leftrightarrow \quad f(\alpha) = -\frac{1}{4\gamma} (\alpha - d - \gamma)^2
\]

Probability Distribution Function

\[
P(\log I_b) \propto \left(\frac{L}{b}\right)^f \left(-\log \frac{I_b}{\log L/b}\right)
\]

Anomalous gIPR

Lognormal PDF
Multifractality of wave function
Faez, Page, Lagendijk and Van Tiggelen

3D diffuse light speckle on 2D surface
3D localized elastic speckle on 2D surface
Localized ultrasound

\[ \Delta_q \approx 0.21 \, q(1-q) \]

is

\[ \gamma \propto \frac{1}{g} \]

\[ ? \]
Observation of 3D ultrasound localization

- Transverse confinement: $D(r)$!
- Anomalous dynamic transmission
- Giant non-poissonian fluctuations $g < 1$
- Multifractal wave function
- $1/t^2$ reflection
- Critical exponent