Mésoscopie des Ondes
Parlons de la phase même……

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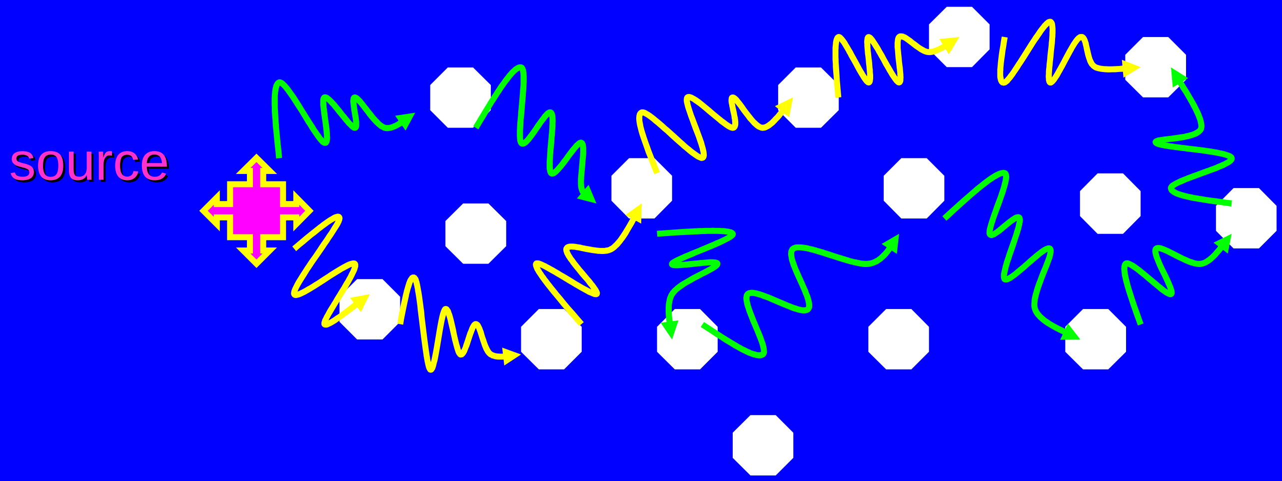
Diffusion Multiple de la Lumière



$\rho(\mathbf{r},t)$ densité d'énergie

$$\partial_t \rho(\mathbf{r},t) - D\ \nabla^2 \rho(\mathbf{r},t) = S\ \delta(t)\ \delta(\mathbf{r}-\mathbf{r}_S)$$

$\int d\mathbf{r}\ \ \rho(\mathbf{r},t) = S$ Conservation globale du flux

$$\langle \mathbf{r}^2(t) \rangle = \frac{\langle \rho(\mathbf{r},t)\mathbf{r}^2 \rangle}{\langle \rho(\mathbf{r},t) \rangle} = 6D\,t$$

$$D = \tfrac{1}{3}\ \mathrm{v}_E\ \ \ell^*$$

Constante de diffusion

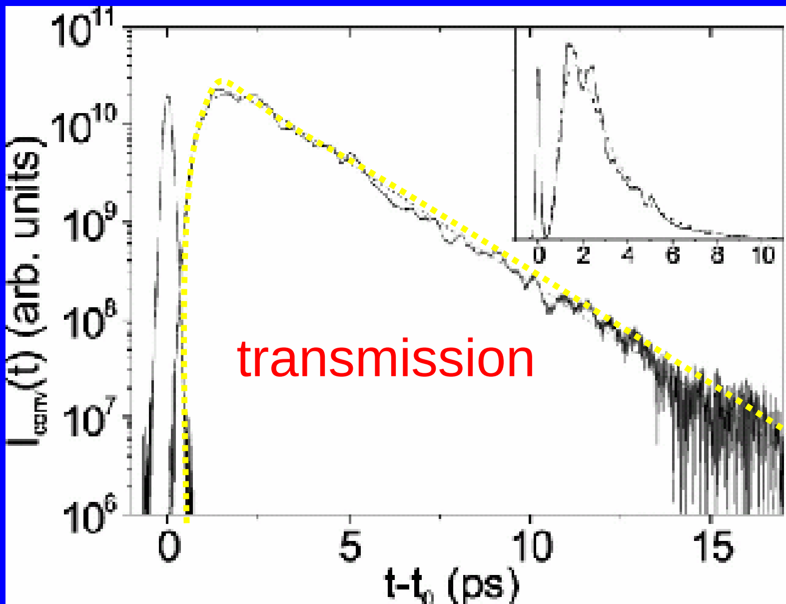
La diffusion des photons, ça marche!

GaP poreux

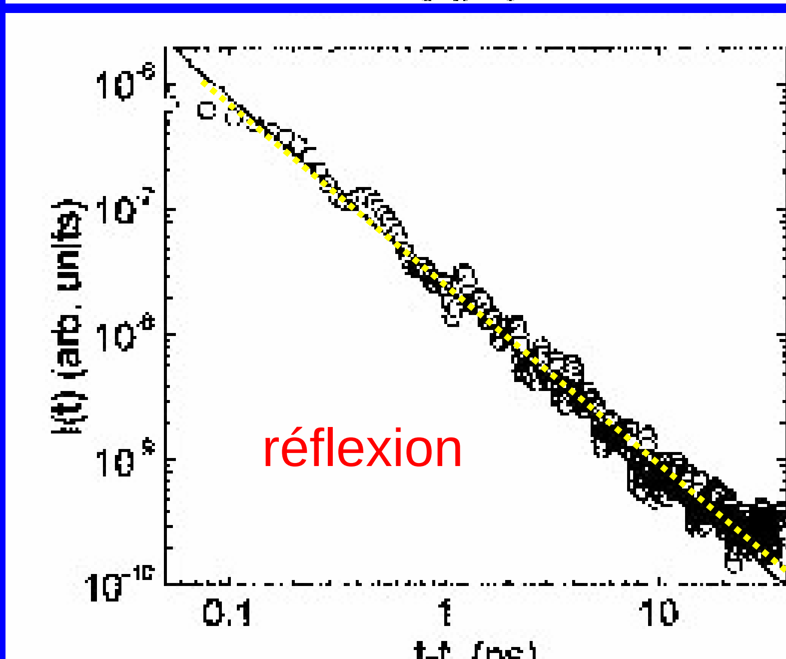
$L = 20\ \mu m$

$\lambda = 739\ nm$

Lagendijk etal, PRE 2003



Équation de diffusion



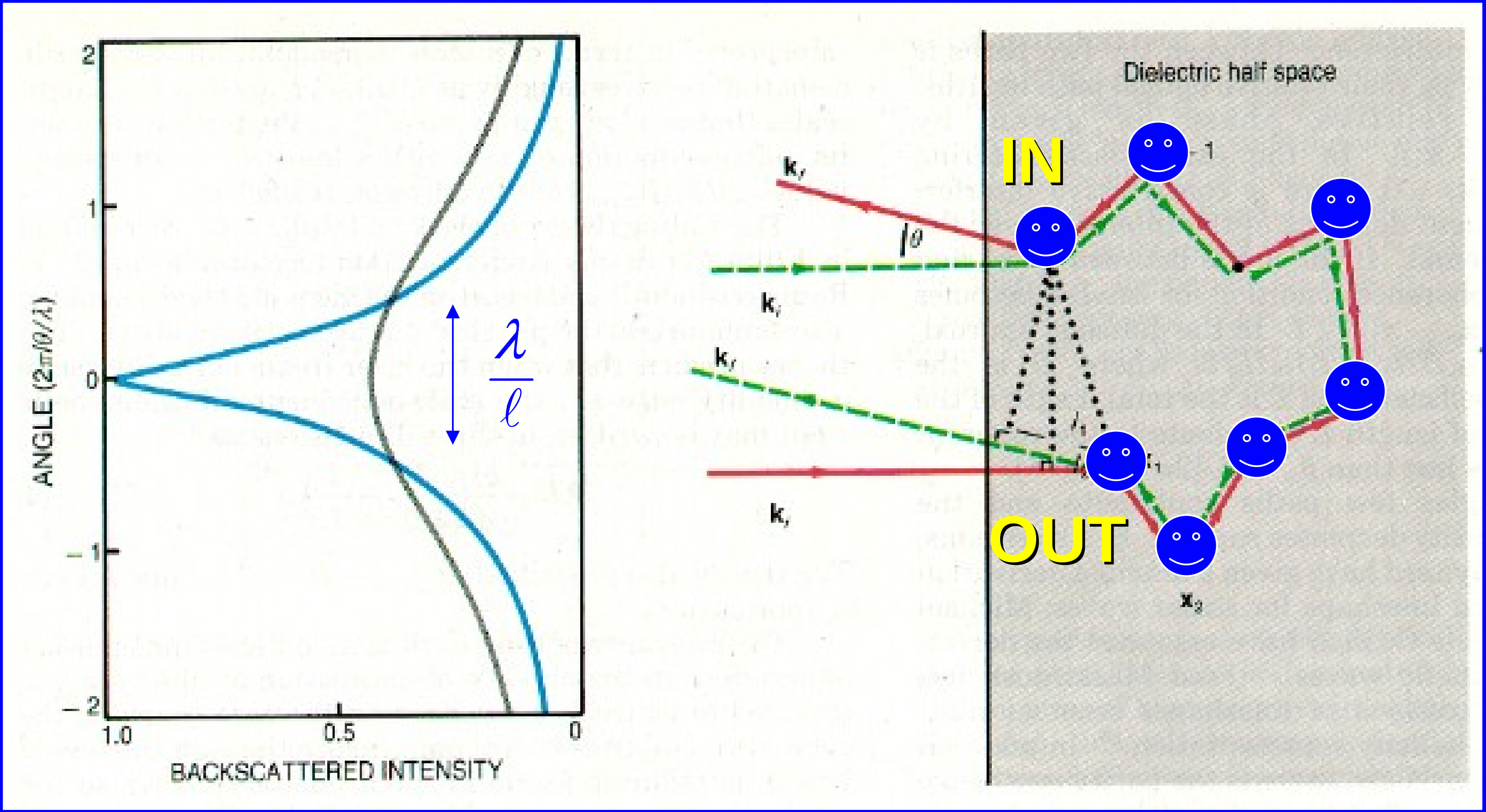
$1/t^{3/2}$

$D = 23\ m^2 / s$

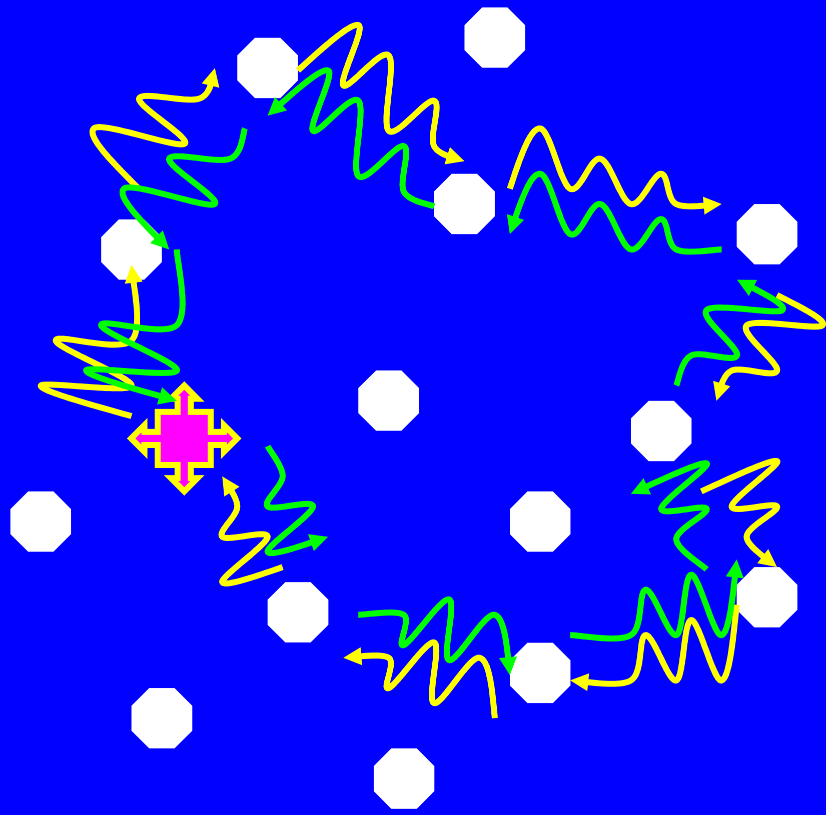
$\ell^* = 250\ nm \quad (k\ell^* = 2.1)$

Coherent Backscattering

Maret & Wolf, Maynard, PRL, 1985
Van Albada & Lagendijk, PRL, 1985



$$\mathrm{CBS} \propto \int_{\mathrm{in,\ out}} \mathrm{G}(\mathrm{IN} \to \mathrm{OUT}) \exp\left[i\mathbf{k}\cdot\left(\mathbf{r}_{\mathrm{in}} - \mathbf{r}_{\mathrm{out}}\right)\right]$$



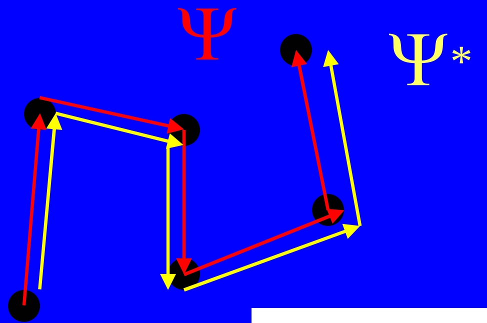
Réciprocité!

Fraction d'énergie qui rentre à la source
au temps *t* par interférence constructive

$$
v_E \, dt \left(\frac{\lambda}{2\pi}\right)^{d-1} \times \rho(\mathbf{r}=0,t) = \begin{cases} \int = \infty & d = 1,2 \\ \int \propto \dfrac{1}{(k\ell)^2} & d = 3 \end{cases}
$$

Description théorique

$$k\ell \gg 1$$

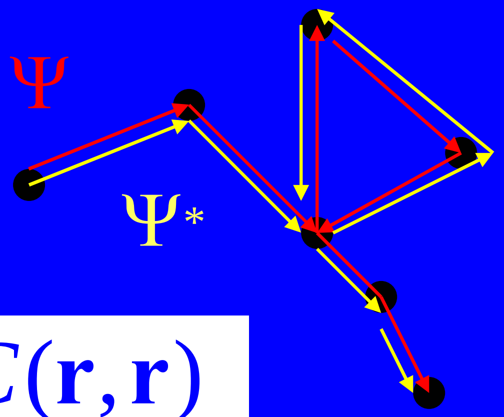


Transfert radiatif

Schwarzshild/Milne , 1900
Chandrasekhar, 1950
Van de Hulst, 1950

$$D_B = \tfrac{1}{3} v_E \, \ell^*$$

$$k\ell \approx 1$$



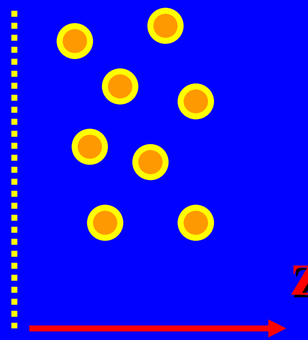
Théorie self-consistente

$$\frac{1}{D(\mathbf{r})} = \frac{1}{D_B} + \frac{C(\mathbf{r},\mathbf{r})}{\pi v_E N(\omega)}$$

Vollhardt & Wölfle, 1980

Dynamique de la Localisation

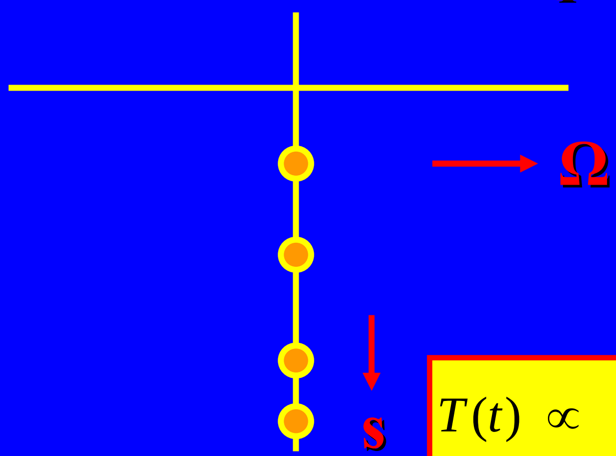
Skipetrov & Van Tiggelen, PRL 2004,2005



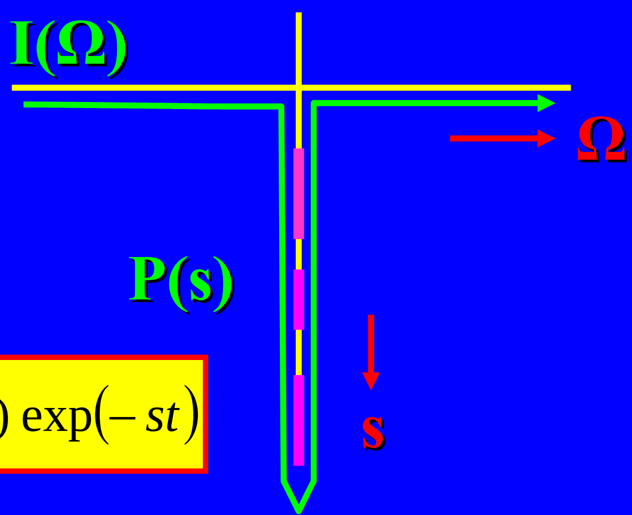
$$-s\, G(z,z',q,s) + \partial_z D(z,s)\, G(z,z',q,s) + q^2 G(z,z',q,s) = \delta(z-z')$$

$$\frac{1}{D(z,s)} = \frac{1}{D_B} + \frac{2}{k^2\ell}\int_{q</3\ell} d^2\mathbf{q}\; G(z,z,q,s)$$

Fréquence complexe Ω+is



$$T(t) \propto \int_0^\infty ds\, G(z_s,z,s)\exp(-st)$$

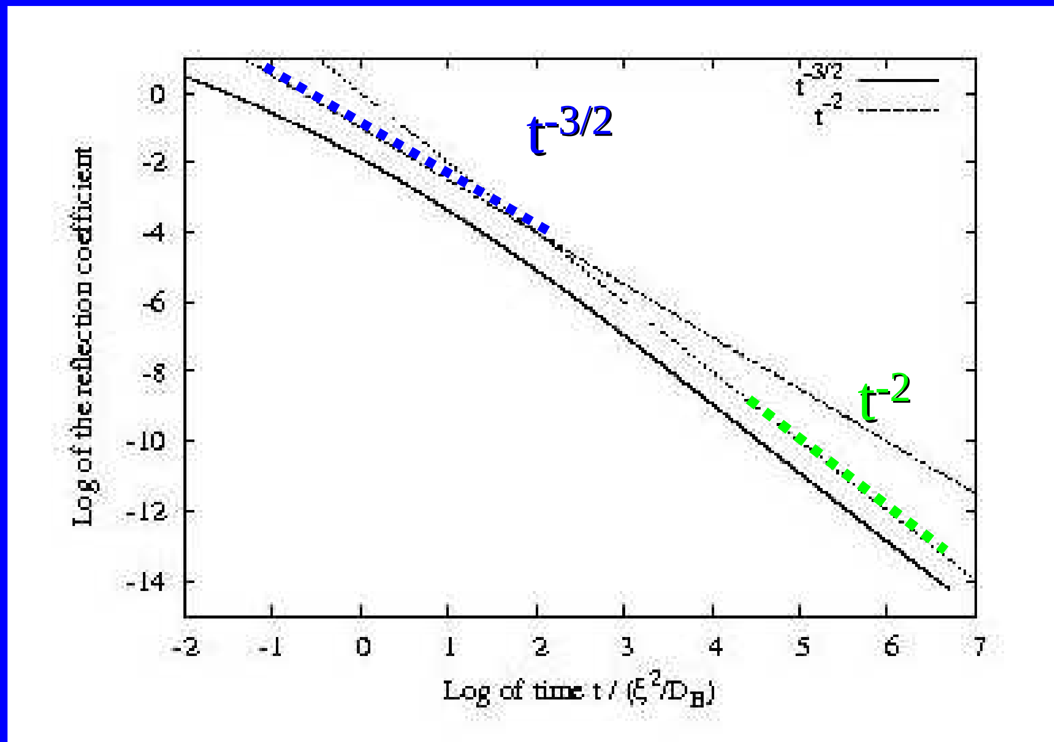


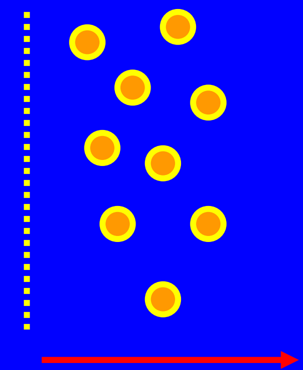
Régime diffus: pôles simples

Régime localisé: coupures de Riemann

3D Demie espace localisée : kℓ=0.7

R(t)





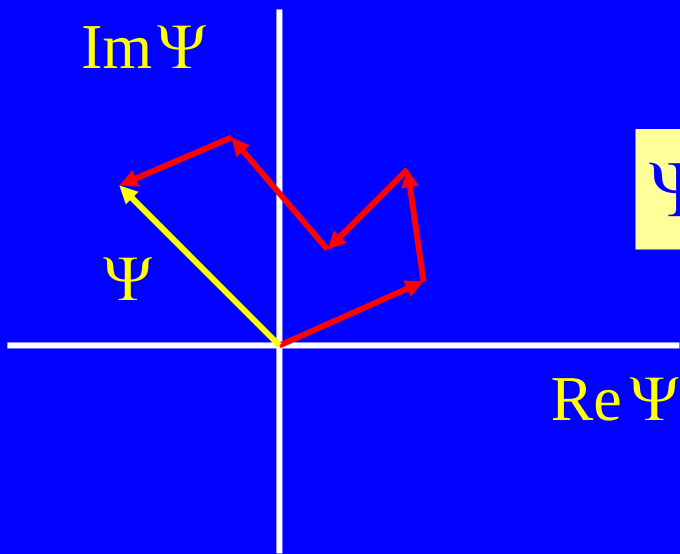
z

time/(ζ^2/D_B)

$$R(t) \propto \frac{1}{t^2}$$

1D sismologie: Burridge, Sheng & Papanicolaou, 1987
Q1D (RMT) Beenakker, 2000

Speckle



$$\Psi = \Psi_1 + \Psi_2 + \Psi_3 + \ldots$$

probability distribution

$$P(\Psi_1, \Psi_2, \ldots \Psi_N) = \frac{1}{\pi^N \det \mathbf{C}} \exp\left(-\mathbf{\Psi}^* \cdot \mathbf{C}^{-1} \cdot \mathbf{\Psi}\right) \qquad C_{ij} \equiv \langle \Psi_i \Psi_j^* \rangle$$

diffusion equation

Gaussian Speckles

$$\Psi = \sqrt{I}\ \mathrm{e}^{\,i\phi}$$

→ intensity

→ phase

1. **Stationary:** Distribution of speckle intensity

$$P(I,\phi) = \frac{1}{<I>}\exp\left(-I/<I>\right)$$

2. **Dynamics :** Distribution of « Wigner delay » time $\dfrac{\mathrm{d}\phi}{\mathrm{d}\omega}$

$$P\left[\Psi\left(\omega - \frac{\Omega}{2}\right),\ \Psi\left(\omega + \frac{\Omega}{2}\right)\right] = \frac{1}{\pi^2 \det \mathbf{C}}\exp\left(-\boldsymbol{\Psi}^* \cdot \mathbf{C}(\Omega)^{-1} \cdot \boldsymbol{\Psi}\right)$$

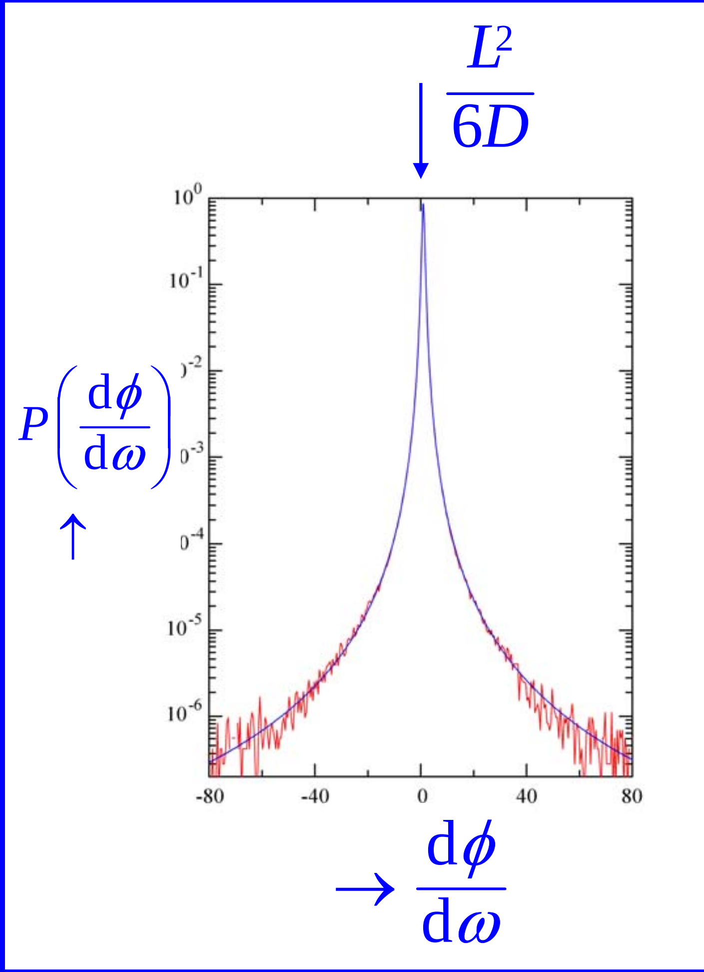
$$\Rightarrow\ P\left(\frac{\mathrm{d}\phi}{\mathrm{d}\omega} = \phi'\right) = \frac{Q}{2}\ \frac{1}{\left[Q + \left(\hat{\phi}' - 1\right)^2\right]^{3/2}}$$

Speckles of Micro-waves in Quasi 1D media

Distribution of delay time in transmission

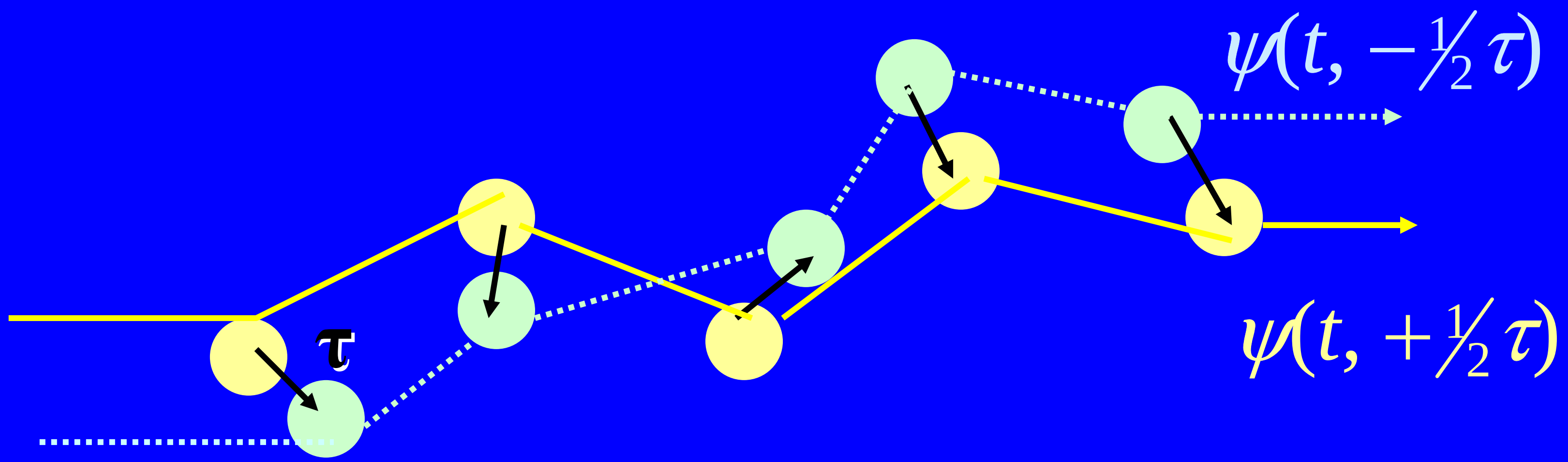
$$P\left(\frac{\mathrm{d}\phi}{\mathrm{d}\omega}=\phi'\right)=\frac{Q}{2}\frac{1}{\left[Q+\left(\hat{\phi}'-1\right)^{2}\right]^{3/2}}$$

diffusion equation : $Q=\frac{2}{5}$



Genack, Sebbah, Stoytchev & Van Tiggelen
PRL, 1999

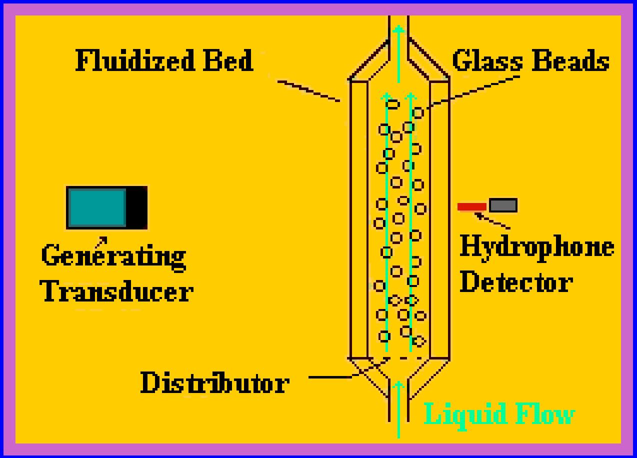
Diffuse Acoustic Wave Spectroscopy



$$n = \frac{ct}{\ell^*}$$

$$\frac{\left\langle \psi\left(t, -\tfrac{1}{2}\tau\right), \psi\left(t, +\tfrac{1}{2}\tau\right)\right\rangle}{\left\langle \psi(t)^2 \right\rangle} = g(\tau) = \exp\left(-\tfrac{1}{6} k^2 n \left\langle \Delta \mathbf{r}^2(\tau) \right\rangle\right)$$

$$g(\tau) \approx \exp\left(-\frac{1}{6}\frac{\tau^2}{t_{\mathrm{DAWS}}^2}\right)$$



Diffuse Acoustic wave Spectroscopy

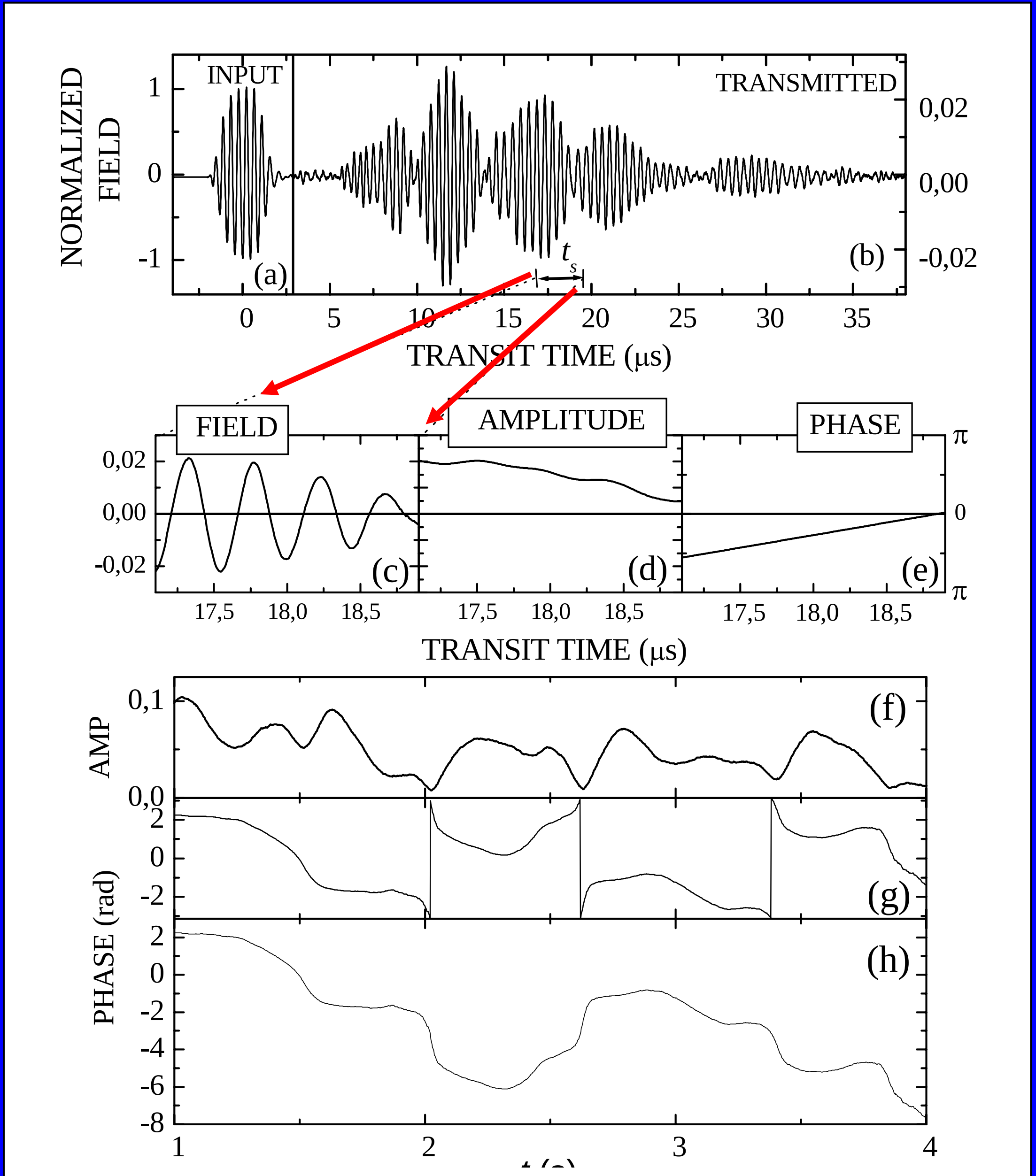
John Page, Dave Weitz, Michael Cowan

amplitude →

Wrapped phase →

unwrapped phase →

$\ell^* = 1.5\, mm; \quad \tau^* = 1\ \mu s$



Time (seconds!)

Probability distribution $P(\Delta\Phi)$

for phase shift $\Delta\Phi(\tau)$

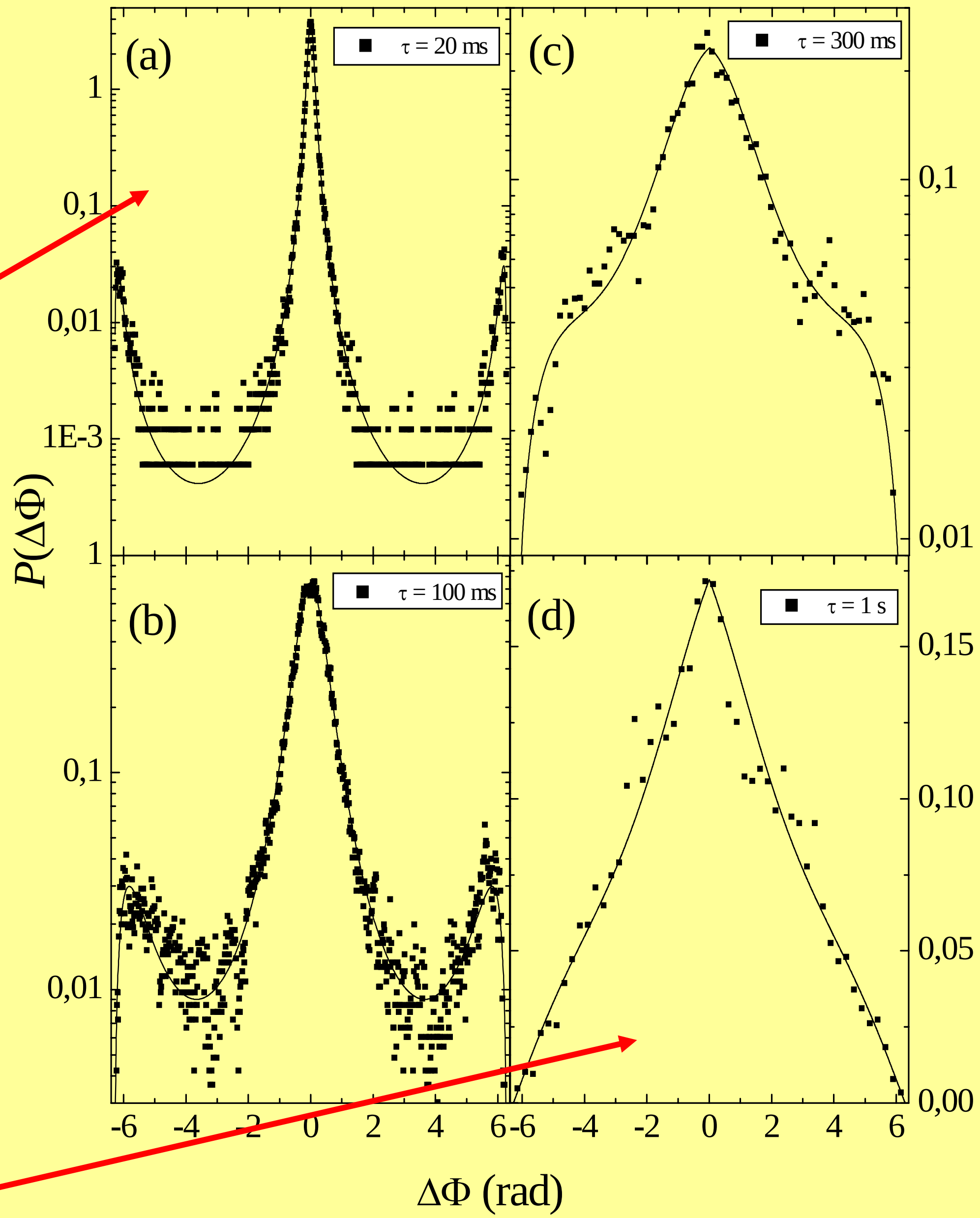
after time τ

$$P\left(\frac{d\phi}{d\tau}\right) = \frac{Q}{\left[2Q + \left(\frac{d\phi}{d\tau}\right)^2\right]^{3/2}}$$

$$Q = \frac{1}{6t_{\mathrm{DAWS}}^2}$$

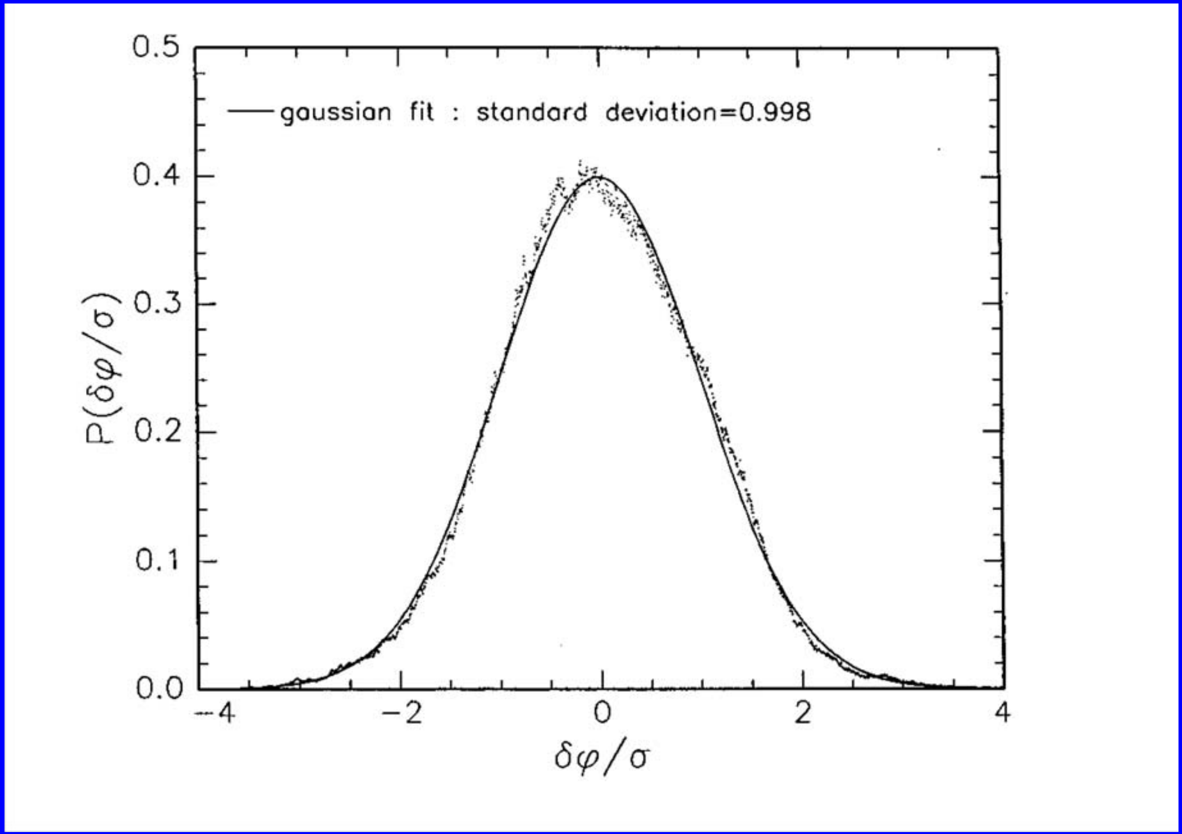
$t_{\mathrm{DAWS}} = 100\,\mathrm{ms}$

$$P(\Delta\phi) = \frac{1}{2\pi^2}\left(2\pi - |\Delta\phi|\right)$$



Cumulative phase $\varphi(t) = \int_0^t d\tau \frac{d\phi}{d\tau}$

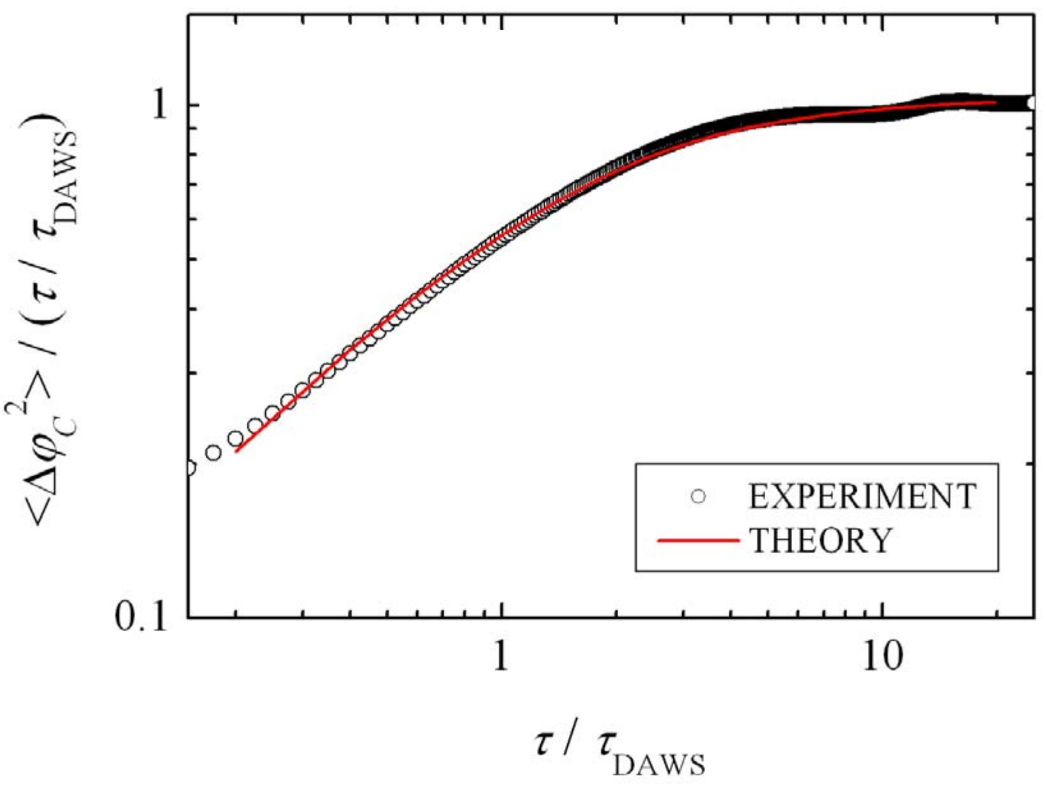
Normal distribution?



Cumulative phase

$$\varphi(t) = \int_0^t d\tau \frac{d\phi}{d\tau}$$

$$\left\langle \varphi^2(t) \right\rangle \xrightarrow{t\to\infty} D_\varphi t$$



Probability distribution of SECOND derivative

$$P\left[\psi(t_1),\psi(t_2),\psi(t_3),\psi(t_4)\right]$$

$$\int\int dA_1\, dA_2\, dA_3\, dA_4\, d\phi_4$$

$$t_4 \rightarrow t_1 \qquad t_3 \rightarrow t_2$$

$$P\left[\phi(t_2)-\phi(t_1),\phi'(t_1),\phi'(t_2)\right]$$

$$\phi(t)=\phi_0+\phi'\Delta t+\frac{1}{2}\phi''(\Delta t)^2+\frac{1}{6}\phi'''(\Delta t)^3$$

?

$$P\left[\phi'(t),\phi''(t),\phi'''(t)\right]$$

Probability distribution of SECOND derivative

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$$P\left[\phi(t_2)-\phi(t_1),\phi'(t_1),\phi'(t_2)\right]$$

$$\phi(t)=\phi_0+\phi'\Delta t+\frac{1}{2}\phi_{\pm}''\left(\Delta t\right)^2$$

$$P\left[\phi'(t),\phi_{+}''(t),\phi''_{-}(t)\right]$$

Phase is not an analytic function

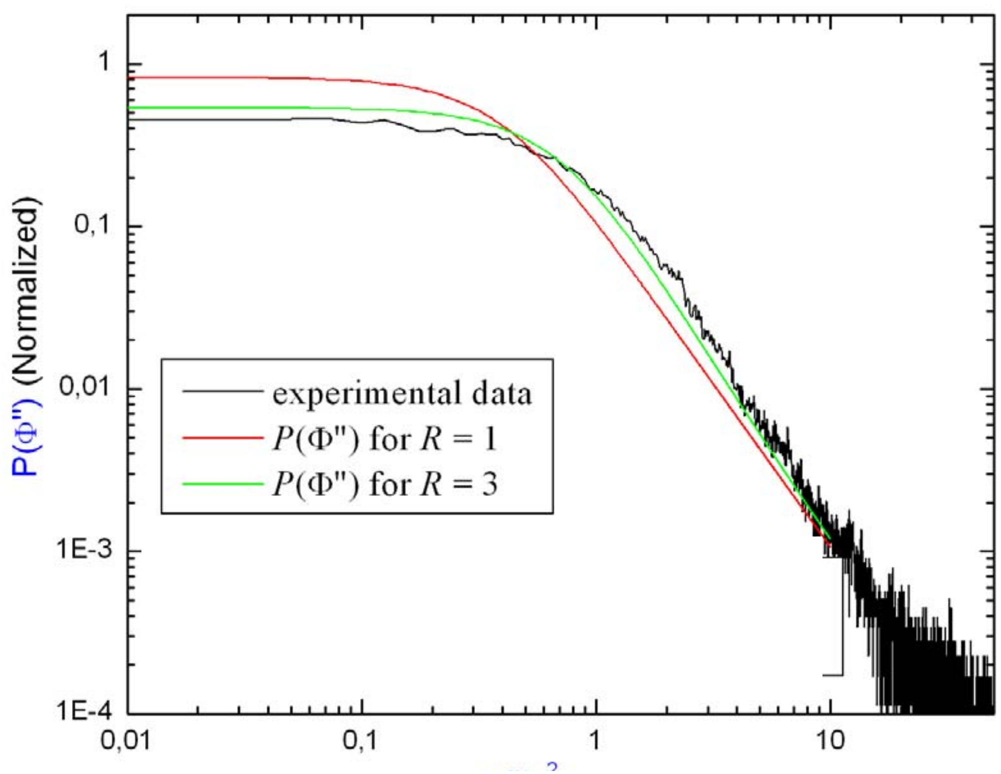
Probability distribution of SECOND derivative

$$P\left[\bar{\phi}''\right] = \frac{1}{\pi}\int_0^\infty dx \frac{\left(4x^2+R\right)^{3/2}}{\left[\left(\bar{\phi}''\right)^2 + \left(x^2+\frac{1}{2}\right)\left(4x^2+R\right)\right]^2}$$

$$P\left[\Delta\phi'' \equiv \phi_+'' - \phi_-''\right] = \frac{1}{4T}\frac{1}{\left[\frac{\left(\Delta\phi''\right)^2}{T}+\frac{1}{2}\right]^{3/2}}$$

$$R = \frac{1}{2}\left[\frac{g^{(4)}(0)}{\left(g''(0)\right)^2} - 1\right] \qquad T = \frac{4}{3}\frac{g^{(4)}(0)}{\left(g''(0)\right)^2}$$

Probability distribution of SECOND derivative



$$\phi'' \times t_{\mathrm{DAWS}}^{2}$$

DAWS signal or dynamic noise ?

Noise is interesting

Spatial Phase

$$\langle \psi(X-\tfrac{1}{2}x)\ \psi^*(X+\tfrac{1}{2}x)\rangle = \frac{\sin kx}{kx}\exp(-x/2\ell)$$

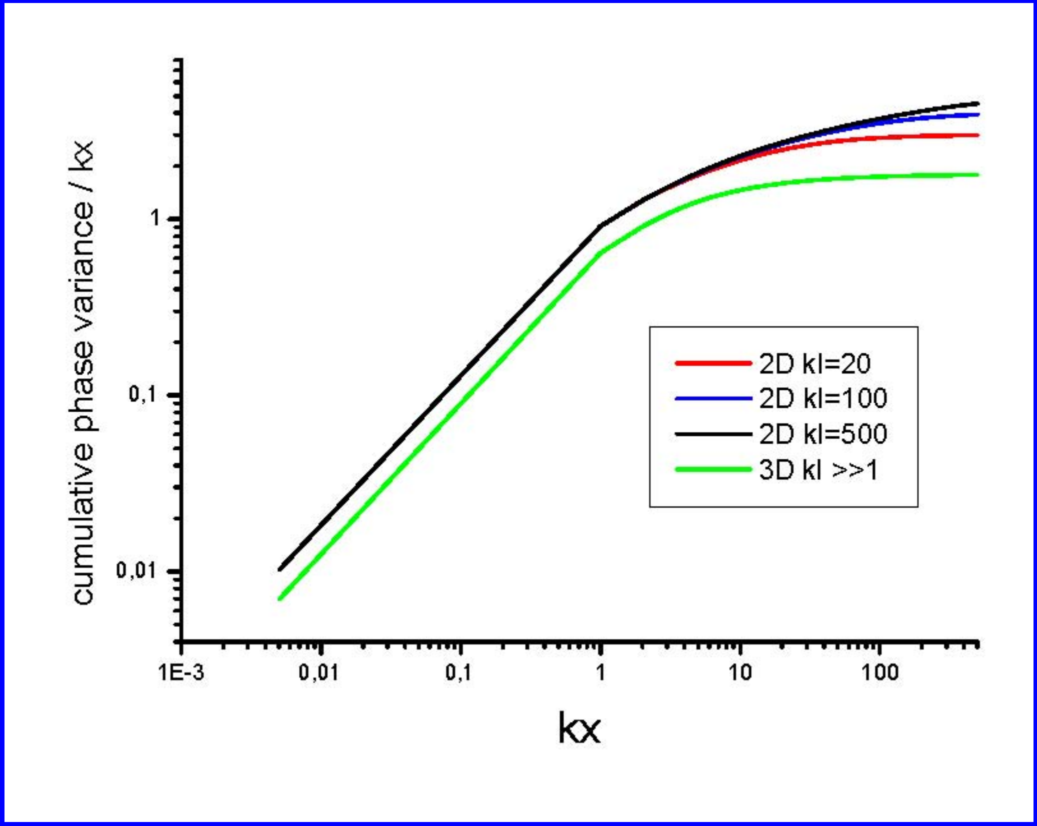
$$C_\phi(x) \equiv \left\langle \frac{d\phi}{dx}(X-\tfrac{1}{2}x)\ \frac{d\phi}{dx}(X+\tfrac{1}{2}x)\right\rangle \xrightarrow{x>\lambda} \frac{1}{x^2}\exp(-x/\ell)$$

Unwrapped phase

$$\varphi(x) = \int_0^x dx' \frac{d\phi}{dx'}$$

variance

$$\langle \varphi^2(x)\rangle = 2\int_0^x dy\ (x-y)C_\phi(y)$$

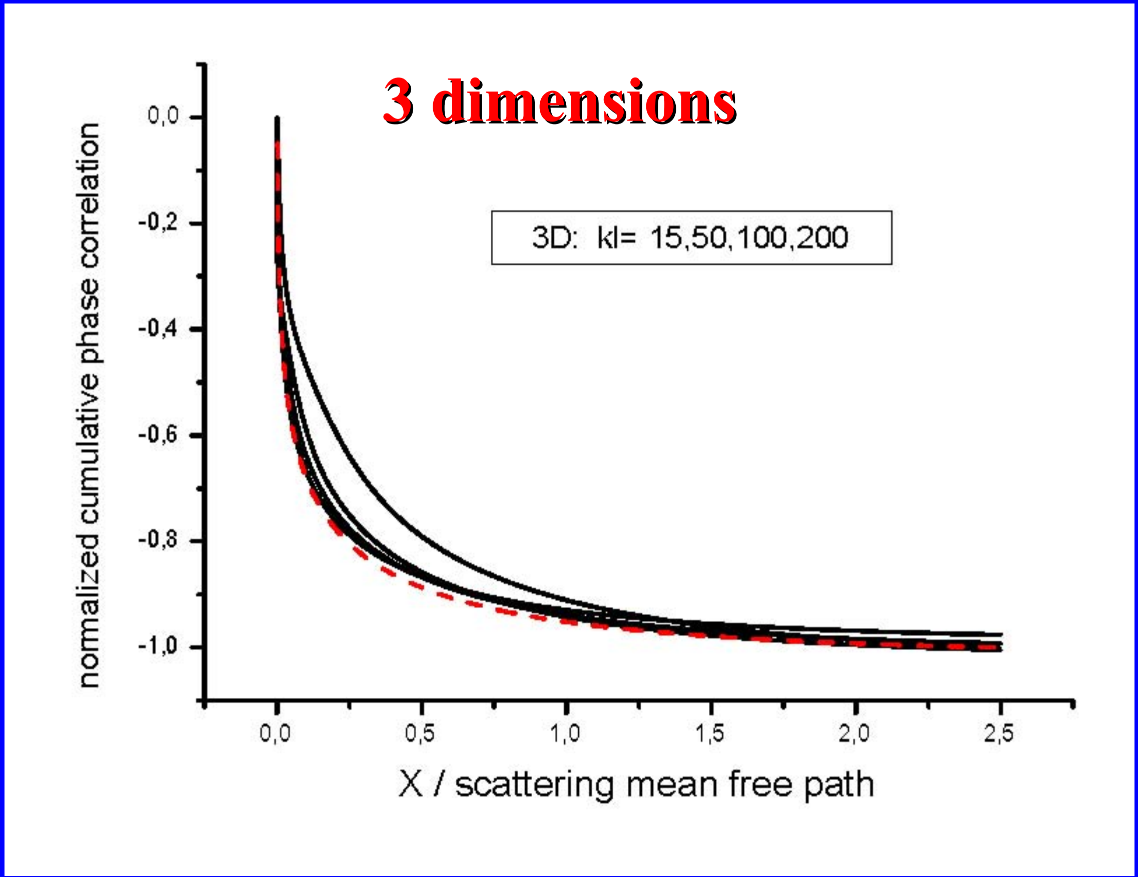


Spatial Phase

$$\langle \psi(X - \tfrac{1}{2}x)\ \psi^*(X + \tfrac{1}{2}x) \rangle = \frac{\sin kx}{kx} \exp(-x/2\ell)$$

$$C_\phi(x) \equiv \left\langle \frac{d\phi}{dx}(X - \tfrac{1}{2}x)\ \frac{d\phi}{dx}(X + \tfrac{1}{2}x) \right\rangle \xrightarrow{x>\lambda} \frac{1}{x^2}\exp(-x/\ell)$$

$$\langle \varphi(-\tfrac{1}{2}x)\varphi(\tfrac{1}{2}x) \rangle$$
$$= -\int_0^{x/2} dy\ y \left[C_\phi(y) + C_\phi(y-x)\right]$$

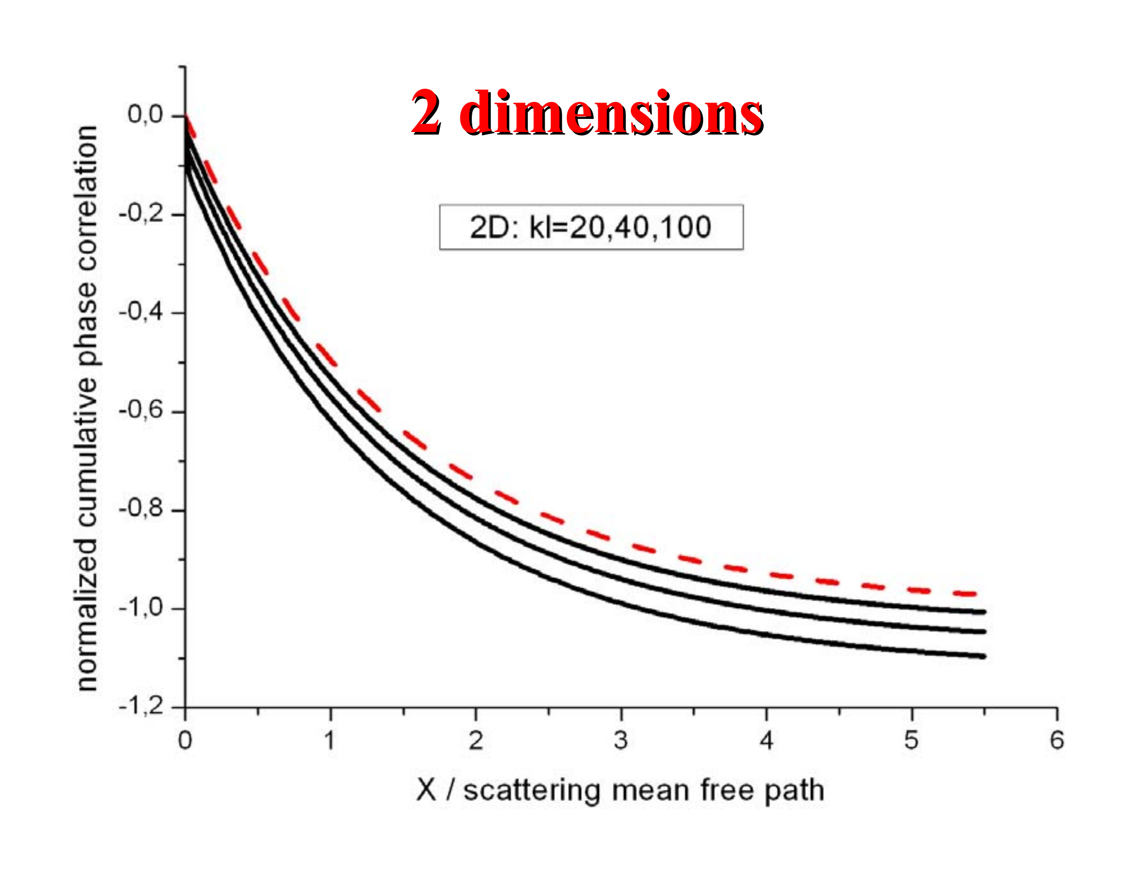


Spatial Phase

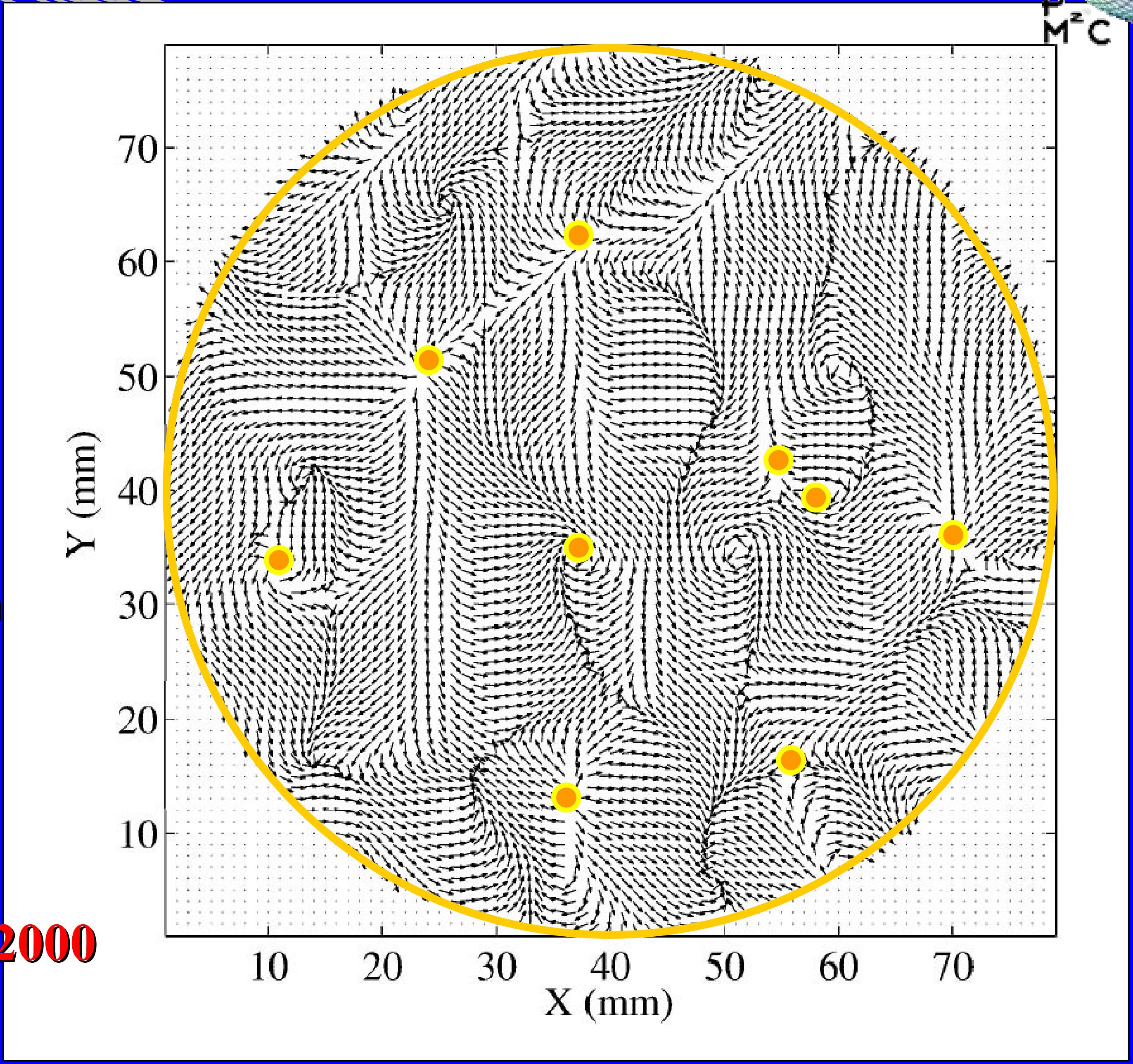
$$\langle \psi(X - \tfrac{1}{2}x)\ \psi^*(X + \tfrac{1}{2}x) \rangle = J_0(kx)\ \exp(-x/2\ell)$$

$$C_\phi(x) \equiv \left\langle \frac{d\phi}{dx}(X - \tfrac{1}{2}x)\ \frac{d\phi}{dx}(X + \tfrac{1}{2}x) \right\rangle \xrightarrow{x>\lambda} \frac{1}{x}\exp(-x/\ell)$$

$$\langle \varphi(-\tfrac{1}{2}x)\varphi(\tfrac{1}{2}x) \rangle = -\int_0^{x/2} dy\ y\left[C_\phi(y) + C_\phi(y-x)\right]$$



Spatial Phase



Patrick Sebbah
Azriel Genack

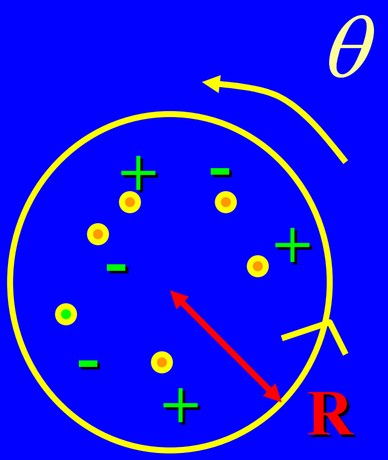
M. Berry, 2000

Spatial Phase

theorem

$$\oint d\mathbf{l}\cdot\nabla\phi(\mathbf{r}) = 2\pi Q$$

$$Q = \sum_{zero\ i} q_i$$



$$\langle Q\rangle = 0$$

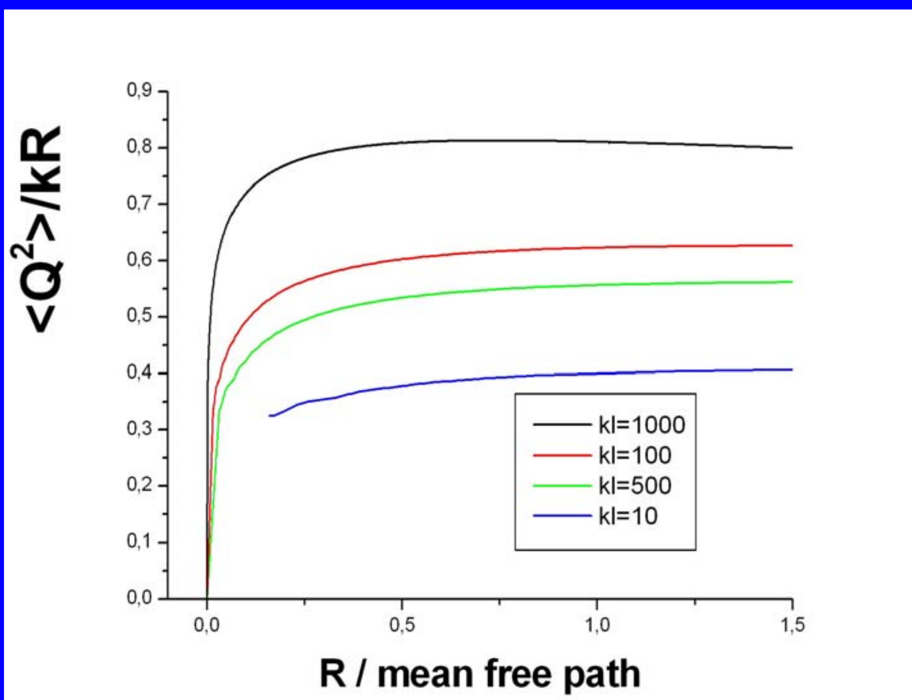
$$\langle Q^2(\text{circle})\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\Delta\theta \left\langle \frac{d\phi}{d\theta}\left(-\frac{\Delta\theta}{2}\right)\frac{d\phi}{d\theta}\left(\frac{\Delta\theta}{2}\right)\right\rangle$$

Count the mean free path?

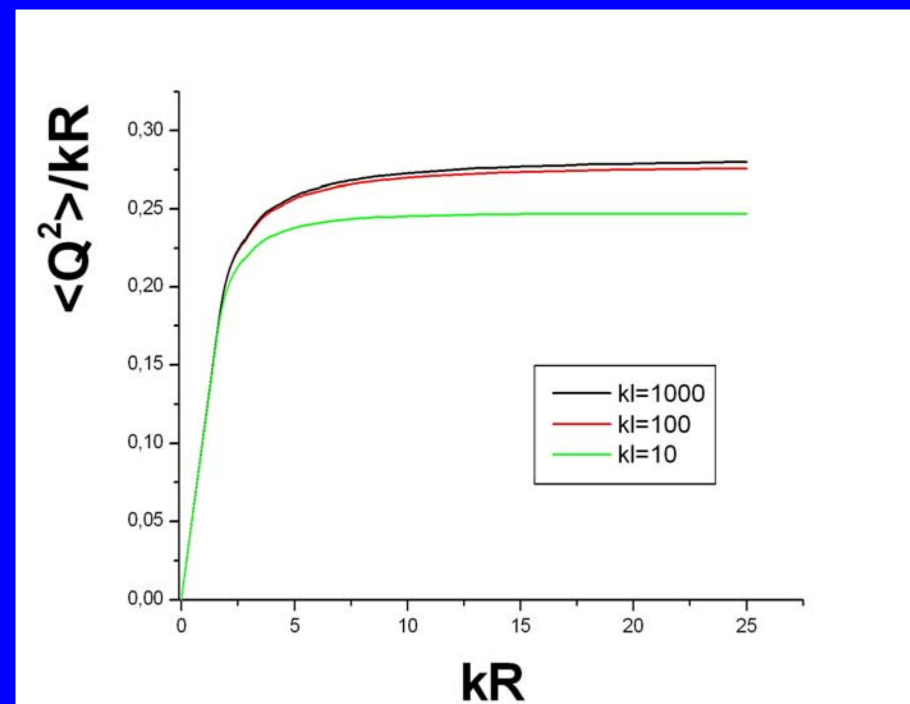
$$\left\langle Q^2(\text{circle})\right\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\Delta\theta \left\langle \frac{d\phi}{d\theta}\left(-\frac{\Delta\theta}{2}\right)\frac{d\phi}{d\theta}\left(\frac{\Delta\theta}{2}\right)\right\rangle$$

$$P\left[\psi(\mathbf{r}_1),\psi(\mathbf{r}_2),\psi(\mathbf{r}_3),\psi(\mathbf{r}_4)\right]$$

$$\left\langle \psi(\mathbf{r})\psi^*(\mathbf{r}')\right\rangle = J_0(k\Delta r)\exp(-\Delta r/2\ell)$$



2 dimensions



3 dimensions

$$\left\langle Q^2 \right\rangle \propto R$$

implies screening of topological charge

(Halperin, 1981, Berry 2000, Wilkinson, 2004)

$$\rho(\mathbf{r}) = \sum_i q_i \ \delta^{(2)}(\mathbf{r} - \mathbf{r}_i)$$

Topological charge density

$$Q = \int_A d^2\mathbf{r} \ \rho(\mathbf{r})$$

Topological charge

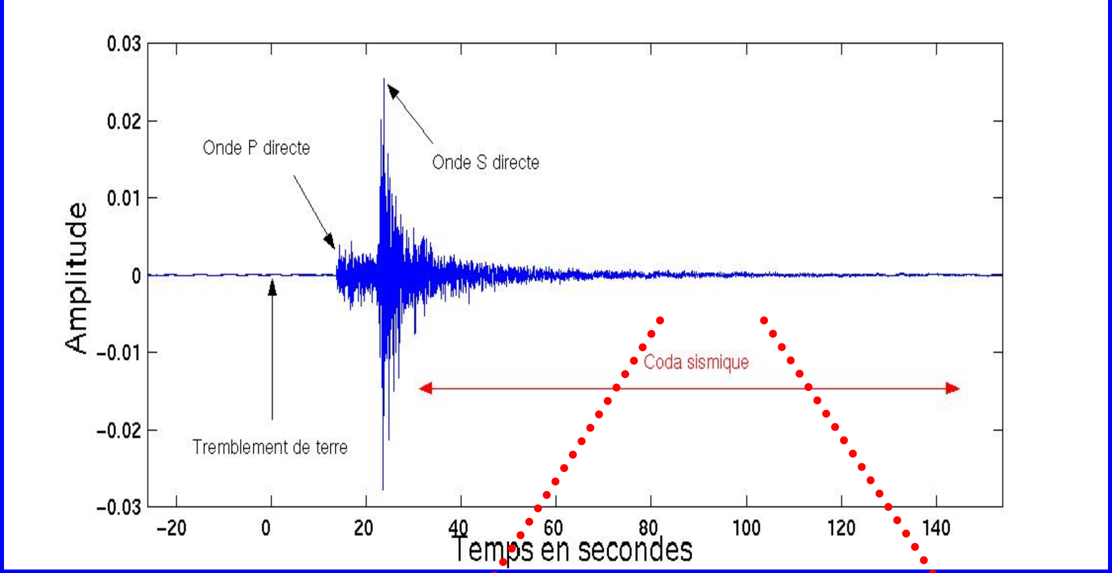
$$C(\mathbf{x}) = \left\langle \rho(\mathbf{r}+\mathbf{x})\rho(\mathbf{r}) \right\rangle$$

Topological pair correlation

$$\left\langle Q^2(R) \right\rangle = \pi R^2 \int_{\Re^2} d^n\mathbf{x} \ C(\mathbf{x}) + \quad (R) \Rightarrow \int_{\Re^2} d^2\mathbf{x} \ C(\mathbf{x}) = 0$$

Imaging without a source

Equipartition → **Correlation = Green function**



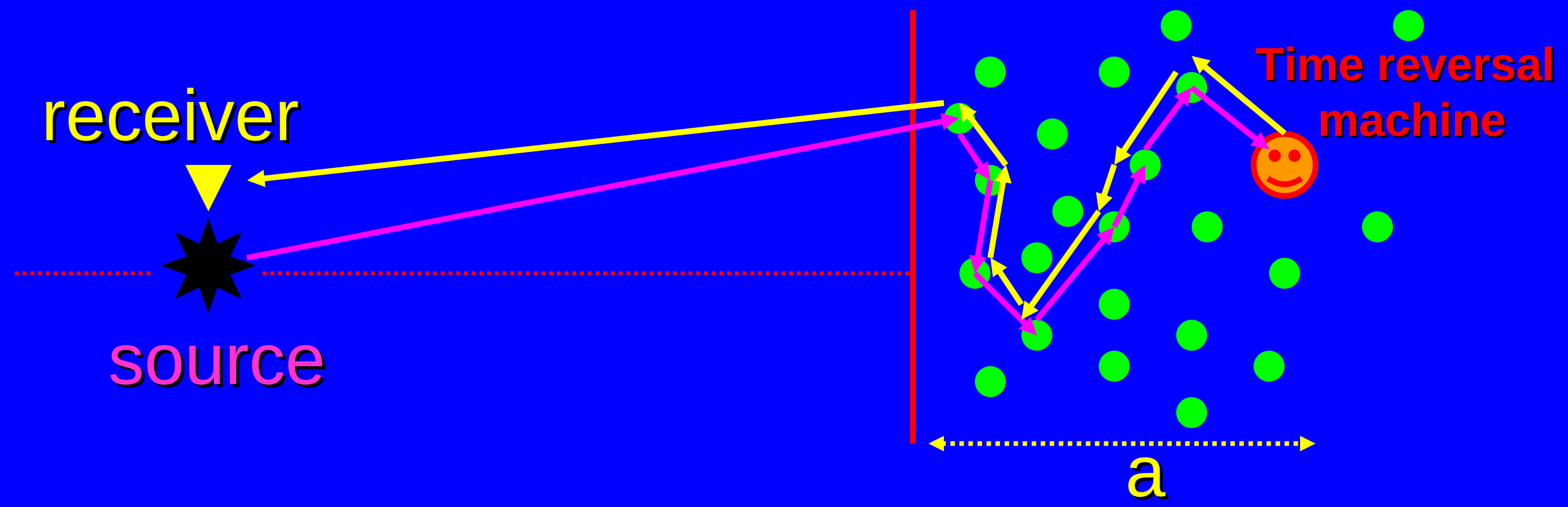
Helio-seismology
Duval, Nature 1993
Thermal phonons
Weaver & Lobkis, PRL 2001
Seismic coda/noise
Campillo etal Science 2003, 2005

$$\left\langle u\left(\mathbf{r}=A, t-\frac{1}{2}\tau\right) u\left(\mathbf{r}=B, t+\frac{1}{2}\tau\right)\right\rangle$$

$$\propto$$

$$G(A \to B, \tau) + G(A \to B, -\tau)$$

Relation with Time-Reversal and Coherent backscattering



$$[S \to TRM \to R](\tau) = \int dt\, [TRM \to S](t-\tau)\, [TRM \to R](t+\tau)$$

Time-reversal ⟷ **correlation method**

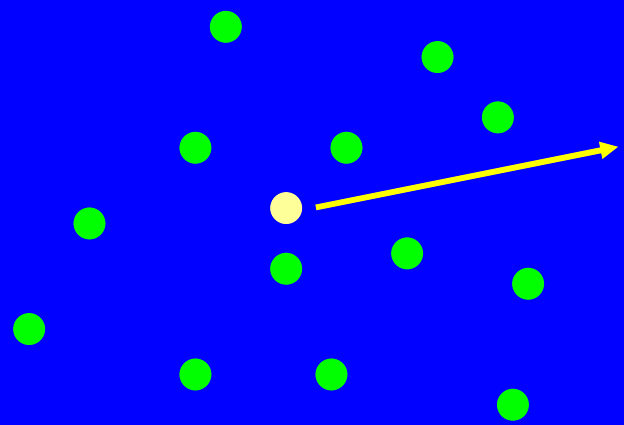
$$R(z,\tau) = S(\tau) \times \mathrm{CBS}\left(\theta \frac{\ell}{\lambda} \to \theta \frac{a}{\lambda}\right) + \text{speckle}$$

$$\approx \sqrt{\frac{D}{Wa^2}} << 1$$

Stable time-reversal at source……

….. with CBS cusp !!

Role of nearby scatterers



$$\rho(\mathbf{r},\omega) = -\frac{\omega}{\pi c}\,\mathrm{Im}\; G(\mathbf{r},\mathbf{r},\omega)$$

Local Density of States

$$\rho(\mathbf{r}) = \lim_{\varepsilon_a \downarrow 0} \frac{2\omega\varepsilon_a}{\pi c} \int d\mathbf{x}\;\; G(\mathbf{r},\mathbf{x})\;\; G^*(\mathbf{x},\mathbf{r})$$

$$\left\langle \rho(\mathbf{r})^2 \right\rangle = \lim_{\varepsilon_a \downarrow 0} \left(\frac{2\omega\varepsilon_a}{\pi c} \right)^2 \int d\mathbf{x}' \int d\mathbf{x}\;\; \left\langle\, G(\mathbf{r},\mathbf{x})\;\; G^*(\mathbf{r},\mathbf{x}) G(\mathbf{r},\mathbf{x}')\;\; G^*(\mathbf{r},\mathbf{x}') \right\rangle$$

$$= \int d\mathbf{x} \;\int d\mathbf{x}'\;\; C_1 + C_2 + C_3 + \ldots\ldots + C_0$$

$$= C_0 \;\approx\; \frac{\pi}{k\ell}$$

(Van Tiggelen & Skipetrov, 2005)