Semi-classical calculations of the correlations

Yves Colin de Verdière
Institut Fourier
Grenoble

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The aim of this lecture is to show how the theory of **pseudo-differential operators (ΨDO's)** builded on the geometry of phase space *(ray theory, Hamiltonian formalism)* can be used in order to compute the high frequency asymptotics of the correlations in the method of **Passive Imaging**.

I will first review the general formula which gives the

field correlation from the source correlation.

I will then give a very brief introduction to ΨDO calculus and apply it to the "semi-classical calculus of the correlations".

I will NOT discuss the physical assumptions!

More details can be found in my paper in *Nonlinearity* **22** (2009).

Other similar contributions by Josselin Garnier & George Papanicolaou.

1. Passive imaging: a general formula for the correlation

2. (A very short introduction to) Semi-classics: (a) Pseudo-differential operators (ΨDO's) (b) Random fields and semi-classics (c) Ray dynamics; (d) Green function; (e) Egorov Theorem.

3. Semi-classical formulas for the correlation

I will discuss the case of a “Schrödinger like” wave equation

$$\frac{du}{dt} + \widehat{H}u = f$$

where u is the field and f the source noise.

- Technically simpler than wave equations: first order time derivative

- The wave equations (acoustical waves, seismic waves) can be decoupled into several “Schrödinger like” equations

$$u_{tt} - Lu = 0 \ / \ v_t = \pm i\sqrt{-L}v$$

1. Passive imaging: a general formula for the correlation

Assuming some source of noise being propagated by a linear wave equation, there is a relation between

- The correlation

  $$C_{A,B}(\tau) = \mathbb{E}\left(u(A,t) \otimes u(B,t-\tau)^\star\right)$$

  ($\mathbb{E} =$ ensemble average or time average) of the fields $u(x,t)$ between 2 points A and B

- The Green function for the wave equation.

Here is the starting point:

$$\frac{d\mathbf{u}}{dt} + \widehat{H}\mathbf{u} = \mathbf{f} \tag{1}$$

- $\mathbf{u}(x,t),\ \ x \in X^d$ the **field** (scalar or vector valued)

- $\widehat{H}$ the **deterministic** smooth (matrix) Hamiltonian, acting on $L^2(X, \mathbb{C}^N)$ includes the **attenuation**:

$$\exists k > 0,\ \ \mathrm{Re} < \widehat{H}\mathbf{u}|\mathbf{u} > \geq k\|\mathbf{u}\|^2$$

- $\mathbf{f}(x,t)$ the **random** source field assumed to be stationary in time and ergodic.

- A model case will be the Schrödinger operator:

$$-i\hbar u_t - \frac{\hbar^2}{2}\Delta u + V(x)u - i\hbar k u = -i\hbar f,\ \ k > 0\ .$$

- A more complicated case will be any kind of wave equation:

$$\mathbf{u}(x,t) := \begin{pmatrix} u \\ u_t \end{pmatrix}$$

and

$$u_{tt} + a u_t - \Delta u = f,\ \ a \geq 0$$

which corresponds to

$$\widehat{H} = \begin{pmatrix} 0 & \mathrm{Id} \\ -\Delta & a \end{pmatrix}$$

and

$$\mathbf{f} := \begin{pmatrix} 0 \\ f \end{pmatrix}$$

The *causal* solution of Equation (1) is given by:

$$\mathbf{u}(x,t) = \int_{-\infty}^{0} ds \int_X P(-s,x,y)\mathbf{f}(t+s,y)|dy| \tag{2}$$

where P, the (time dependent) Green function, is defined as follows:

P is the integral kernel of $\Omega(t) = \exp(-t\widehat{H})$

$$(\Omega(t)\mathbf{v})(x) = \int_X P(t,x,y)\mathbf{v}(y)|dy| \ .$$

In what follows, we will denote $[A](x,y)$ the integral kernel of the operator A.

$\Omega(t+s) = \Omega(t) \circ \Omega(s)$ rewrites

$$\int_X P(t,x,y)P(s,y,z)|dy| = P(t+s,x,z)$$

Correlation of the field

We define the correlation matrix

$$C_{A,B}(\tau) := \lim_{T\to+\infty} \frac{1}{T}\int_0^T \mathbf{u}(A,t)\otimes\mathbf{u}^\star(B,t-\tau)dt$$

or

$$C^{\alpha\beta}_{A,B}(\tau) := \lim_{T\to+\infty} \frac{1}{T}\int_0^T \mathbf{u}_\alpha(A,t)\overline{\mathbf{u}_\beta(B,t-\tau)}dt$$

Putting $\mathbf{u}(A,t)$, $\mathbf{u}(B,t-\tau)$ as given by Equation (2), we get:

$$C_{A,B}(\tau) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \Phi(T_t f)dt \tag{3}$$

with

$$\Phi(f) = \int_{-\infty}^0 ds \int_{-\infty}^0 ds' \int_{X\times X} |dxdy| \cdots$$
$$\cdots P(-s,A,x)\mathbf{f}(x,s)\otimes(P(-s',B,y)\mathbf{f}(y,s'-\tau))^\star$$

Assuming the correlation of the source given by

$$\mathbb{E}(\mathbf{f}(x,s)\otimes \mathbf{f}^{\star}(y,s')) = \delta(s-s')K(x,y)$$

and ergodicity, we get, for $\tau > 0$:

$$C_{A,B}(\tau) = \int_{-\infty}^{0} ds \int_{X^2} |dx||dy| P(\tau - s, A, x) K(x,y) (P(-s,B,y))^{\star} \tag{4}$$

and $C_{A,B}^{\alpha\beta}(-\tau) = \overline{C_{B,A}^{\beta\alpha}(\tau)}$.

For $\tau > 0$, we can rewrite Equation (4) in an operator form:

$$C_{A,B}(\tau) = [\Omega(\tau)\Pi](A, B) \tag{5}$$

with

$$\Pi := \int_0^\infty \Omega(s)\mathcal{K}\Omega^\star(s)ds \tag{6}$$

where $\mathcal{K}$ is the integral operator whose kernel is $K(x, y)$ (the correlation of the source). This is the completely general relation between the correlation and the Green function.

The semi-classical asymptotics of the Green function is well known *(ray theory, Van Vleck formula, Fourier Integral Operators)*. We will try to compute Π in the semi-classical regime.

We are lead to the following problem: find the high frequency behavior of $\Omega(\mathrm{s})\mathcal{K}\Omega^{\star}(\mathrm{s})$ under some appropriate assumptions on $\mathcal{K}$

If f is a white noise, i.e. $\mathcal{K} = \mathrm{Id}$, we have

$$C_{A,B}(\tau) = [\Omega(\tau) \int_0^\infty \Omega(s)\Omega^\star(s) ds](A, B)$$

If we assume $\hat{H} = \widehat{H_0} + k\mathrm{Id}$ with $\widehat{H_0}$ self-adjoint, we get

$$C_{A,B}(\tau) = \frac{e^{-2k|\tau|}}{2k} P(\tau, A, B) \tag{7}$$

In general, i.e. for non homogeneous noises, Equation (7) is only valid approximately!

Case of the wave equation:

$$u_{tt} + 2ku_t - \Delta u = f \tag{8}$$

Let $Q = \sqrt{-\Delta - k^2}$ and $G(t,x,y)$ the integral kernel of $\frac{\sin tQ}{Q}$.

We get

$$u(x,t) = \int_0^\infty e^{-ks} ds \int_X G(s,x,y) f(y,t-s)|dy|$$

And the correlation, for $\tau > 0$, in case of a white noise,

$$C_{A,B}(\tau) = \frac{e^{-k\tau}}{4(Q^2+k^2)} \left[\frac{\cos \tau Q}{k} + \frac{\sin \tau Q}{Q} \right] (A,B)$$

The τ derivative of $C_{A,B}(\tau)$ is

$$-\frac{e^{-k\tau}}{4k} G(\tau, A, B)$$

2. Semi-classics

We want a nice class of operators for which we can study the high frequency limits of $\Omega(s)B\Omega^\star(s)$.

They are called the *pseudo-differential operators* (ΨDO's) and were introduced in the sixties by Calderon, Zygmund, Nirenberg, Hörmander as a tool in the study of linear partial differential equations with non constant coefficients.

In some sense, they give the geometrical extension of Hamiltonian formalism of *classical mechanics* to *wave mechanics*.

In applications to physics, it is often called the **ray** method. The same tools apply to the study of the semi-classical limit of quantum mechanics and to the high frequency limit of wave equations (acoustic, electromagnetic or seismic waves).

There is a **small parameter** $\varepsilon > 0$ in the theory which can be Planck "constant" $\hbar$ or the inverse of the frequency ω^{-1}.

- (a) ΨDO's

- (b) Random fields: power spectra and correlations.

- (c) Ray dynamics

- (d) Semi-classical Green functions

- (e) Egorov Theorem

(a) ΨDO's

ε will be a **small parameter**: in what follows

- $\varepsilon \sim$ inverse of the frequency
- $\varepsilon \sim$ typical correlation distance of the noisy source, i.e. $K(x, y) = k(x, y, \frac{x-y}{\varepsilon})$

A pseudo-differential operator (ΨDO) on $\mathbb{R}^d$

$$A_\varepsilon := \mathrm{Op}_\varepsilon(a)$$

is defined using a function $a(x,\xi) : \mathbb{R}^d \oplus \mathbb{R}^d \to \mathbb{C}$ (a is called the *symbol*) on the phase space. a is assumed to be

- smooth

- homogeneous near infinity in ξ

$$A_\varepsilon(f)(x) = \frac{1}{(2\pi)^d} \int e^{i(x-y|\xi)} a\left(\frac{x+y}{2}, \varepsilon\xi\right) f(y) |dy d\xi|$$

Simple examples:

- $\mathrm{Op}_\varepsilon(\xi_j) = \frac{\varepsilon}{i}\frac{\partial}{\partial x_j}$

- $\mathrm{Op}_\varepsilon(x_j)$ is the multiplication by x_j

- $\mathrm{Op}_\varepsilon(\chi(\xi))$ is a frequency cut-off

- $\mathrm{Op}_\varepsilon(|\xi|^2 + V(x)) = -\varepsilon^2\Delta + V(x)$: a Schrödinger operator

Pseudo-differential operators act nicely on WKB functions:

$$\mathsf{Op}_\varepsilon(a)(A(x)e^{iS(x)/\varepsilon}) \approx a(x, S'(x))A(x)e^{iS(x)/\varepsilon}$$

The integral kernel of $\mathsf{Op}_\varepsilon(a)$ is

$$\varepsilon^{-d}\widehat{a}\left(\frac{x+y}{2}, \frac{x-y}{\varepsilon}\right)$$

where $\widehat{a}(x, X)$ is the partial Fourier transform w.r. to ξ of $a(x,\xi)$.

The main properties are the following ones which hold as $\varepsilon \to 0$:

- Composition:

$$\mathsf{Op}_\varepsilon(a) \circ \mathsf{Op}_\varepsilon(b) \approx \mathsf{Op}_\varepsilon(ab)$$

- Brackets:

$$[\mathsf{Op}_\varepsilon(a), \mathsf{Op}_\varepsilon(b)] \approx \frac{\varepsilon}{i}\mathsf{Op}_\varepsilon\{a, b\}$$

  where

$$\{a, b\} = \sum_{j=1}^{d} \left(\frac{\partial a}{\partial \xi_j}\frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j}\frac{\partial b}{\partial \xi_j} \right)$$

  is the Poisson bracket

Wigner functions:

Wigner functions define the localization of energy in phase space. The Wigner function W_u of u is the function on the phase space defined by

$$\int aW_u|dxd\xi| =< \mathsf{Op}_\varepsilon(a)u|u > \quad ,$$

or

$$W_u(x,\xi) = \frac{1}{(2\pi)^d}\int e^{-iv\xi}u\left(x+\frac{\varepsilon v}{2}\right)\bar{u}\left(x-\frac{\varepsilon v}{2}\right)|dv| \ .$$

We have

$$\int W_u(x,\xi)|d\xi| = |u|^2, \ \int W_u(x,\xi)|dx| = |\mathcal{F}u|^2 \ .$$

(b) Random fields: power spectra and correlations

Let $f = f(x)$ be a random field.

1) The **correlation**

$$C(x, y) := \mathbb{E}(f(x)\bar{f}(y))$$

2) The **power spectrum**

$$P := \mathbb{E}(W_f)$$

is a function on the phase space.

P and C are related by:

C(x,y) is $((2\pi\varepsilon)^d$ times) **the operator kernel of Op(P).**

P is $((2\pi\varepsilon)^{-d}$ times) **the symbol of the operator C**

C and P carry the same information.

Ex1: the **white noise**

$C = \delta(x - y)$, $P = 1/(2\pi\varepsilon)^d$.

Ex2: a **stationary noise** on $\mathbb{R}$ with $\varepsilon = 1$

$C(s, t) = F(s - t)$ and $P(s, \omega)$ is the Fourier transform $\mathcal{F}(F)(\omega)$.

(c) Ray dynamics

If $H(x, \xi)$ is the Hamiltonian function, the associated ray dynamics is defined by the vector field X_H:

$$\begin{cases} \frac{dx_j}{dt} & = \frac{\partial H}{\partial \xi_j} \\ \frac{d\xi_j}{dt} & = -\frac{\partial H}{\partial x_j} \end{cases}$$

If $H = \frac{1}{2}\|\xi\|^2 + V$, we get Newton equations. If $H = \frac{1}{2}g^{ij}\xi_i\xi_j$, we get the geodesics.

We will denote by ϕ_t the flow of X_H:

$$\frac{d}{dt}(\phi_t(z)) = X_H(\phi_t(z))$$

(d) Green function

Let us assume that our wave dynamics, $\Omega(\tau) = \exp(-\tau\widehat{H})$, is generated by $\widehat{H} = \frac{i}{\varepsilon}\mathrm{Op}_\varepsilon H$. What is the semi-classical behavior of P?

$P(\tau, x, y)$ is a sum of contributions from rays going from y to x in time τ.

Van Vleck formula

$$P(\tau, A, B) \sim \sum_{\gamma \in R^{\tau}_{AB}} P_{\gamma} \tag{9}$$

with R^{τ}_{AB} the set of rays from B to A in time τ.

In the generic case (non caustic points)

$$P_{\gamma} \sim a_{\gamma}(\varepsilon) e^{\frac{i}{\varepsilon} S(\gamma)}$$

where S is the Lagrangian action $S(\gamma) = \int_0^{\tau} (\xi dx - H dt)$.

Let us remark that as a function of A and B, S is a generating function of the flow at time τ.

Formally, as it is well known, vV formulas can be derived from Feynman path integral, by applying stationary phase:

$$(FPI)\ P(\tau, A, B) = \int_{\Omega^{\tau}_{AB}} e^{iS(\gamma)/h} d\gamma$$

where

- Ω^{τ}_{AB} is the set of paths in the configuration space from B to A

- $d\gamma$ is a (mathematically ill defined) measure on Ω^{τ}_{AB}

- $S(\gamma) = \int_0^{\tau} \mathcal{L}(\gamma(t), \dot{\gamma}(t))dt$ is the (Lagrangian) action integral.

Non caustic condition is equivalent to non degeneracy of the Hessian of S.

(e) Egorov Theorem

Let us consider the scalar case with no attenuation ($\widehat{H}$ is self-adjoint, unitary dynamics) $\Omega(t) = \exp(-it\widehat{H}/\varepsilon)$ with $\widehat{H} = \mathrm{Op}_\varepsilon(H)$.

Théorème 1 (Egorov, 70's) *If $A = \mathrm{Op}_\varepsilon(a)$,*

$$A_t := \Omega(-t)A\Omega(t) \approx \mathrm{Op}_\varepsilon(a \circ \phi_t)$$

where ϕ_t is the Hamiltonian flow of H.

Proof:

it is enough to look at the derivative, say at $t = 0$:

$$\frac{d}{dt}_{|t=0} A_t = \frac{i}{\varepsilon}[\widehat{H}, A]$$

and by the ΨDO calculus:

$$\frac{d}{dt}_{|t=0} A_t \approx \mathsf{Op}_\varepsilon\{H, a\}$$

and remember

$$\{H, a\} = X_H a \ (= \frac{d}{dt}_{|t=0}(a \circ \phi_t) \ ) \ .$$

3. A semi-classical formula for the correlation

We will assume that the power spectrum of the random source $f(x,t)$ is $p(x,\xi)$ (with $p \geq 0$), ie

$$C(x,y,s,t) = k\left(\frac{x+y}{2}, \frac{x-y}{\varepsilon}\right)\delta(s-t)$$

where k is the partial Fourier transform of p w.r.t. to ξ.

Applying the previous tools, we want to compute the leading terms in the behavior of $C_{A,B}(\tau)$:

we get

$$C_{A,B}(\tau) = [\Omega(\tau)\Pi](A,B)$$

where Π is a ΨDO whose symbol $\pi(x,\xi)$ can be explicitly computed as a (convergent) integral over the trajectories (rays) arriving at the point x.

More precisely, if such a phase space trajectory γ satisfies $\gamma(0) = (x, \xi)$, $\pi(x, \xi)$ is an integral over the $t < 0$ part of γ. This integral is non-vanishing if the $t < 0$ part of γ crosses the support of the power spectrum p of f.

From Egorov Theorem, assuming for simplicity a constant attenuation k, the symbol $\pi \geq 0$ of

$$\Pi = \int_0^\infty \Omega(s)\mathcal{K}\Omega^\star(s)ds$$

is given by

$$\pi(x,\xi) = \int_{-\infty}^0 p(\phi_t(x,\xi))e^{2kt}dt$$

with p the power spectrum of the source noise.

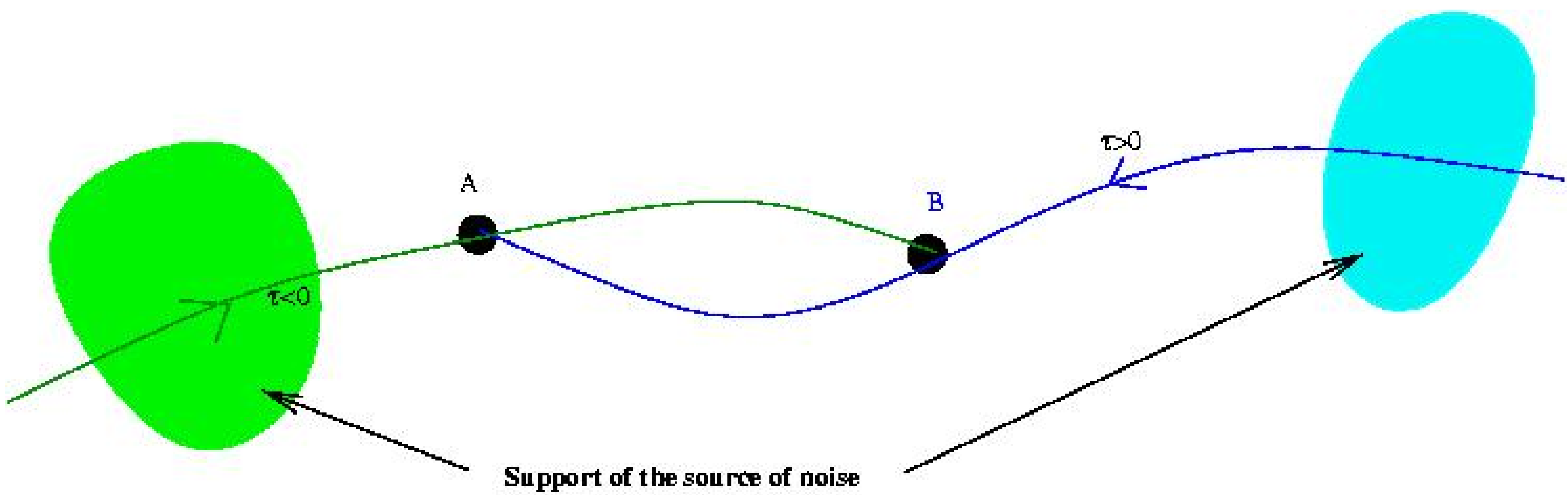
Using Van Vleck formula, we get the main result

$$C_{A,B}(\tau) = \sum_{\gamma} \pi(B, (\xi_B)_{\gamma}) P_{\gamma}$$

(and in the generic case

$$C_{A,B}(\tau) = \sum_{\gamma} \pi(B, (\xi_B)_{\gamma}) a_{\gamma}(\varepsilon) e^{iS(\gamma)/\varepsilon} \ )$$

with $(\xi_B)_{\gamma}$ the value of the impulsion at B of the trajectory γ.


A
B
τ<0
τ>0
Support of the source of noise

We see that this is very close to the Green function

$$P(\tau, A, B) = \sum_{\gamma} P_{\gamma}$$

- Same phases for the contribution of each trajectory
- Modification of the amplitude in terms of the ray dynamics, the power spectrum of the source and the attenuation