Point processes for the study of multiple scattering signals

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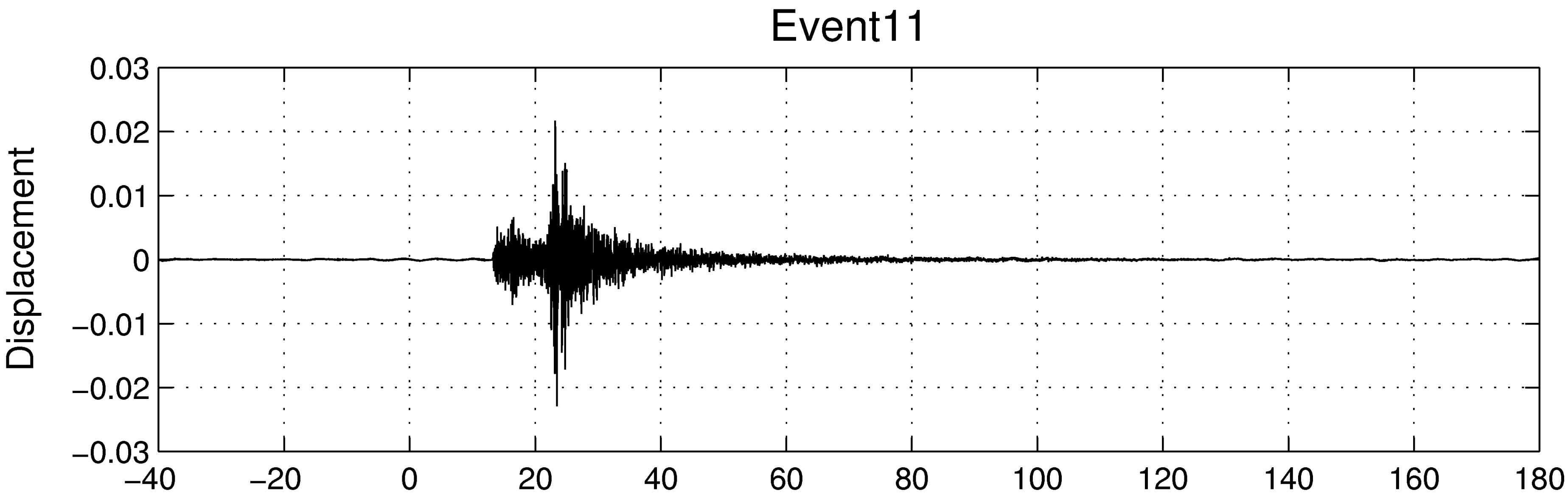
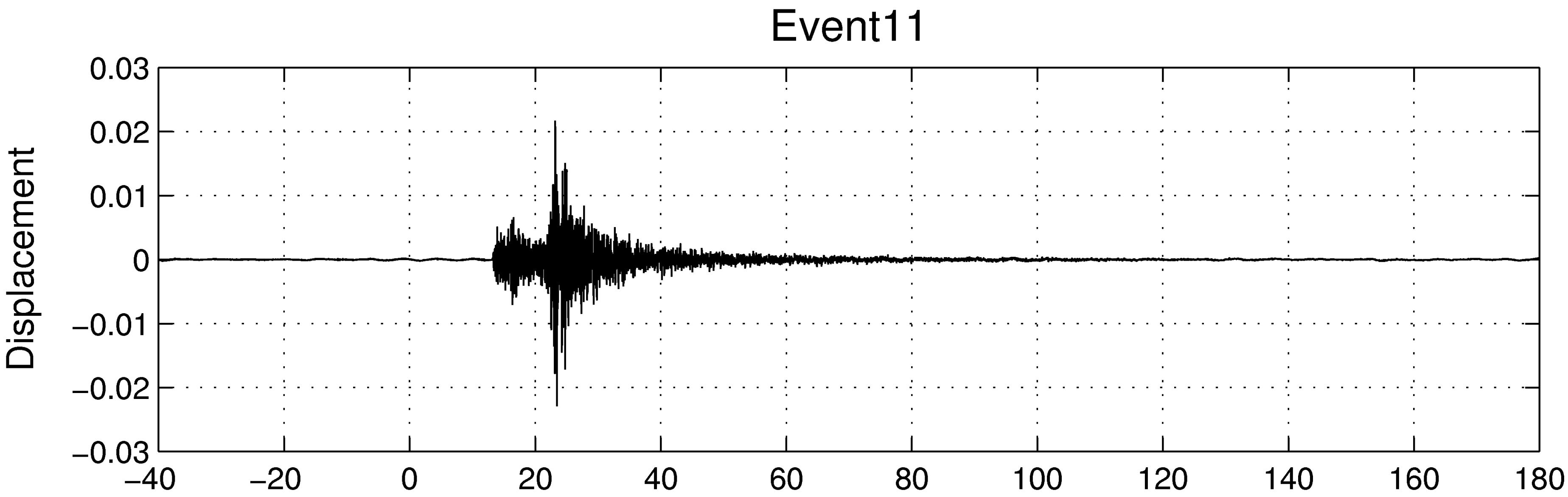
Coda waves



Coda waves $\neq$ deterministic description of waves

$\Rightarrow$ Multiple scattering, equipartition theory to study coda waves

Coda waves



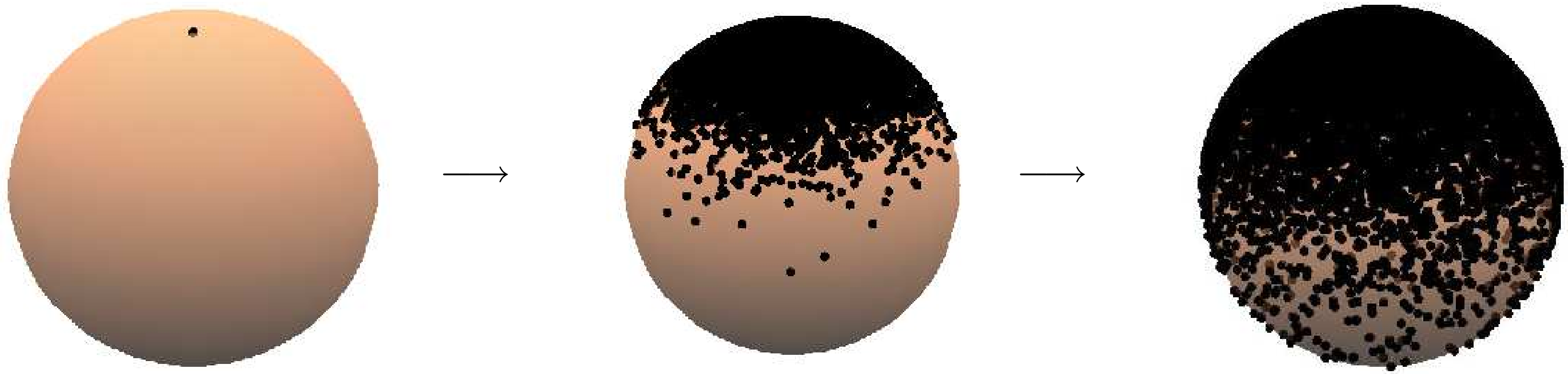
Coda waves $\neq$ deterministic description of waves

$\Rightarrow$ Multiple scattering, equipartition theory to study coda waves

$\Rightarrow$ **Random signal model**

Depolarization effect

Polarization state *spreading* on Poincaré sphere



Propagation of polarized light through optical fiber with birefringence.

Depolarization effect

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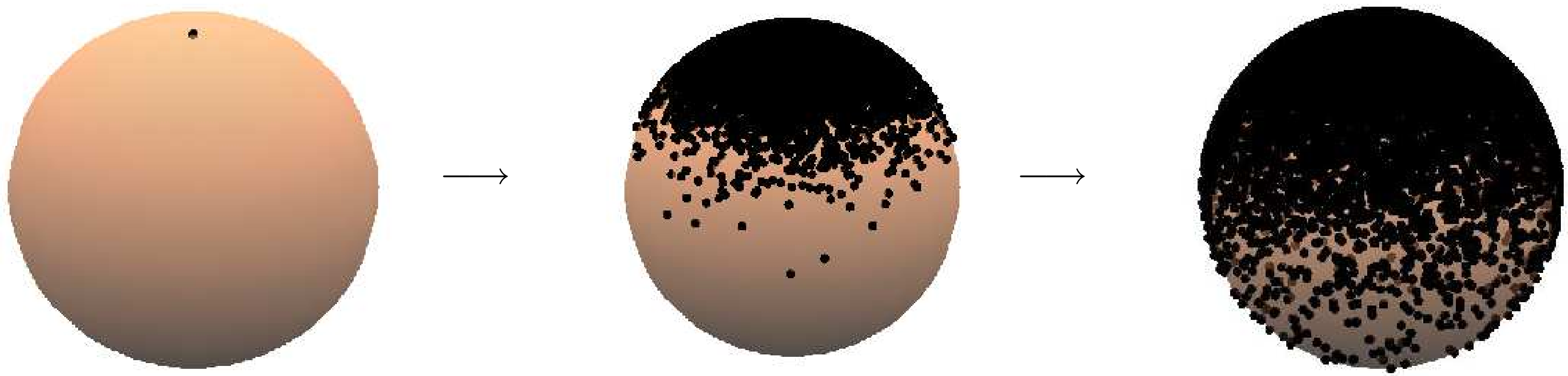

Propagation of polarized light through optical fiber with birefringence.

Birefringence $\Rightarrow \prod$ of random rotations of the polarization state

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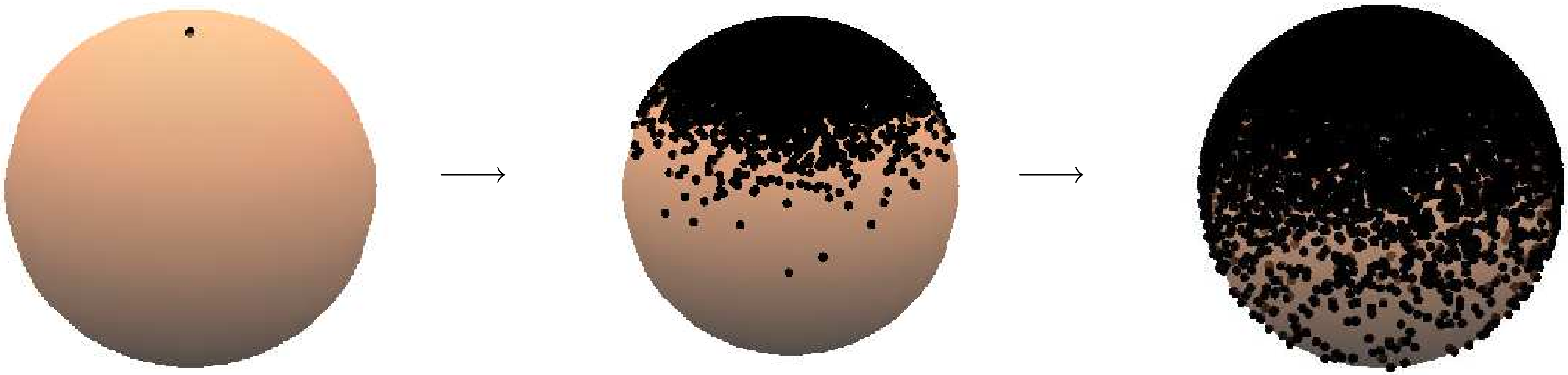


Propagation of polarized light through optical fiber with birefringence.

Birefringence $\Rightarrow \prod$ of random rotations of the polarization state

$\Rightarrow$ Statistics on $\mathcal{S}^2$ and $SO(3)$, and noncommutative harmonics

Stochastic processes models & tools

- Point processes and Lévy processes
- Harmonic analysis on compact Lie groups
- Stochastic differential equations (paths observation)
- Estimation theory
- Inverse problem (statistical inference)

Stochastic processes and random media: Existing work

⋆ **Physics**

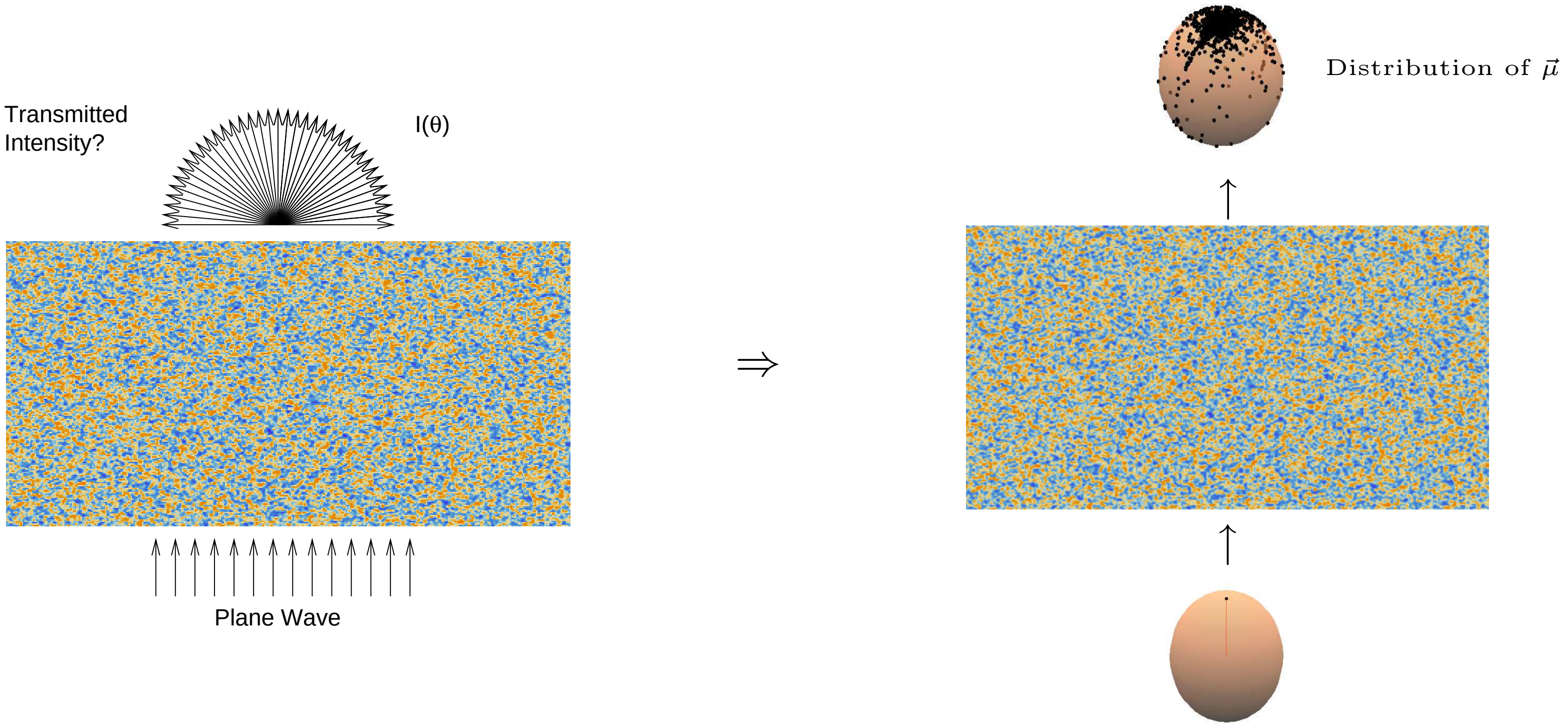
- Compound Poisson process (CPP) for forward scattering (*Ning et al.*, PRE 95)
  ↪ Real-valued CPP + direct problem
- Lévy processes & depolarization (*Said et al.*, WCRM 08)
  ↪ Noncommutative harmonic analysis

⋆ **Signal, Information & Statistics**

- Communication & random media (*Franceschetti et al.*, IEEE & JOSA 04,06,07)
  ↪ 2D Random walk + percolation
- Information transfert & random media (*Skipetrov*, PRE 03)
  ↪ Channel capacity + multiple scattering
- Statistical inference & multiple scattering (*Le Bihan et al.*, PRE 09)
  ↪ CPP on Lie groups + random media characterisation

Forward multiple scattering

- Distribution of intensity $I(\theta)$ and direction of propagation $\vec{\mu}$



Forward multiple scattering

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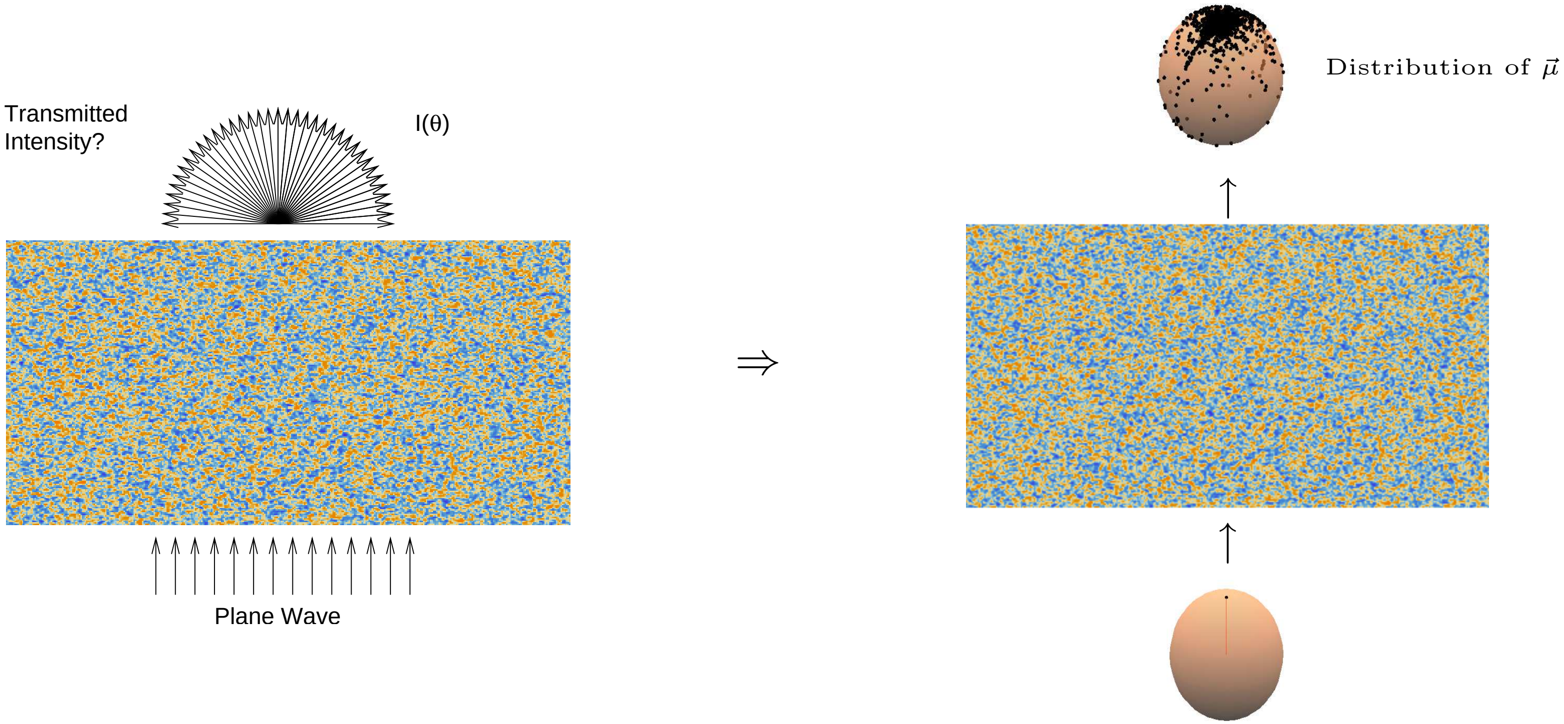


$\hookrightarrow$ Stochastic process model for $\vec{\mu}$

Outline

0 - ~~Introduction~~

1 - **Compound Poisson Processes (CPP)**

2 - CPP and multiple scattering signals

3 - Decompounding and estimation of the phase function

4 - CPP and the geometric phase

5 - Conclusions

Compound Poisson process on $\mathbb{R}$

$\star$ Definition

The random process $y(t)$ defined as the random sum:

$$y(t) = \sum_{i=1}^{N(t)} x_i$$

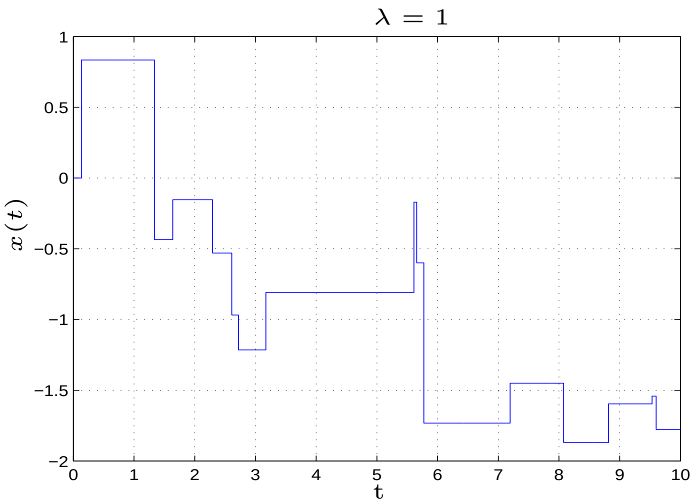
where x_i are i.i.d. real valued random variables and $N(t)$ is a Poisson process (parameter λ) independent of x_i, is called a `Compound Poisson Process`.

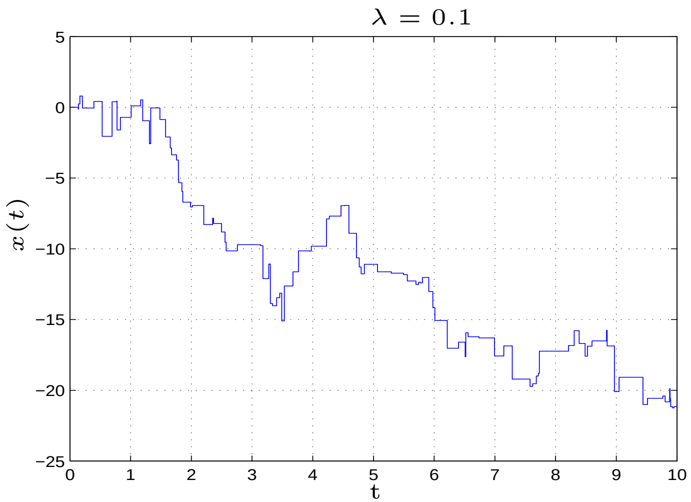
$\star$ Some properties

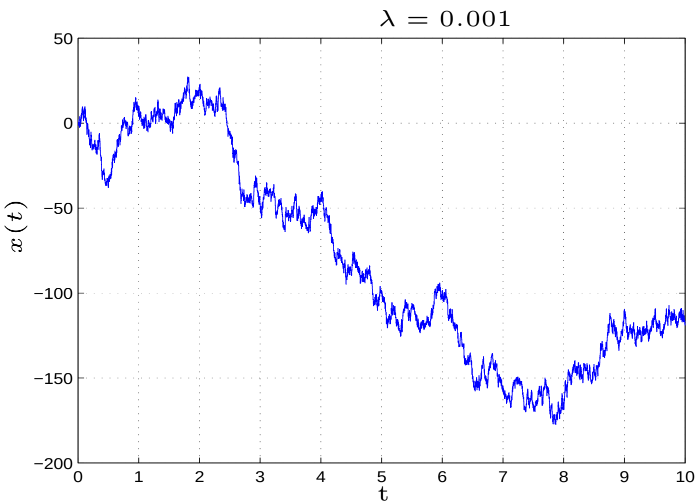
- Mean: $\mathbb{E}[y(t)] = m_x \lambda t$
- Variance: $Var[y(t)] = (\sigma_x^2 + m_x^2)\lambda t$
- Characteristic function: $\Phi_{y(t)}(u) = \mathbb{E}[e^{uy(t)}] = \exp\left(\lambda t(\Phi_x(u) - 1)\right)$

where $\sigma_x^2 = \mathbb{E}[(x_1 - \mathbb{E}[x_1])^2]$ and $\Phi_x(u) = \mathbb{E}[e^{ux_1}]$

⋆ <u>Examples of sample paths</u>







⋆ <u>Note</u>: $y(t) \approx \mathcal{N}\left(m_x \lambda t, (\sigma_x^2 + m_x^2)\lambda t\right)$ when $t \to +\infty$

Compound Poisson process on $SO(3)$

⋆ Definition

The random process $Y(t)$ defined as the random product:

$$Y(t) = \prod_{i=1}^{N(t)} X_i$$

where X_i are i.i.d. $SO(3)$-valued random variables and $N(t)$ is a Poisson process (parameter λ) independent of x_i, is called a `Compound Poisson Process`.

⋆ Characteristic function

$$\Phi_{Y(t)}(l) = \exp\left(\lambda t(\Phi_x(l) - I_{2l+1})\right)$$

where I_{2l+1} is the $(2l+1) \times (2l+1)$ identity matrix and exp is the matrix exponential.

Characteristic function for $SO(3)$-valued random variables

⋆ Peter-Weyl theorem on $SO(3)$

Any function $f \in L^2(SO(3), \mathbb{C})$ with respect to the Haar measure on $SO(3)$ has a Fourier expansion given by:

$$f(\phi, \theta, \psi) = \sum_{l \geq 0} \sum_{m=-l}^{l} \sum_{n=-l}^{l} (2l+1) \hat{f}^l_{mn} \overline{D^l_{mn}(\phi, \theta, \psi)}$$

where ZXZ convention is used for Euler angles (ϕ, θ, ψ), and where the Wigner-D functions $D^l_{mn}(\phi, \theta, \psi)$ are given by:

$$D^l_{mn}(\phi, \theta, \psi) = e^{\mathrm{i}m\phi} P^l_{mn}(\cos\theta) e^{\mathrm{i}n\psi}$$

⋆ $SO(3)$-valued random variables

If f is the *pdf* of a $SO(3)$-valued random variable $\Rightarrow$ its characteristic function is the set of $(2l+1) \times (2l+1)$ matrices $\hat{f}^l_{mn}$ given by:

$$\hat{f}^l_{mn} = \int_{SO(3)} f(\phi, \theta, \psi) D^l_{mn}(\phi, \theta, \psi) dg(\phi, \theta, \psi)$$

with $dg(\phi, \theta, \psi)$ the Haar measure on $SO(3)$.

Compound Poisson Process on $SO(3)$ observed on $\mathcal{S}^2$

Consider a unit vector $\mu(t) \in \mathcal{S}^2$ consisting in the transitive action of a CPP on $SO(3)$ on an initial vector μ_0:

$$\mu(t) = \prod_{i=1}^{N(t)} X_i \mu_0$$

$\star$ Characteristic function

$$\Phi_{\mu(t)}(l) = \exp\left(\lambda t(\Phi_x(l)\Phi_{\mu_0}(l) - I_{2l+1})\right)$$

where

- $\Phi_{\mu(t)}(l)$ are $(2l+1)$ vectors
- $\Phi_{\mu_0}(l)$ are $(2l+1)$ vectors
- $\Phi_x(l)$ are $(2l+1) \times (2l+1)$ matrices.

CPP on $SO(3)$ observed on $\mathcal{S}^2$

⋆ Examples of sample paths

CPP on $\mathcal{S}^2$ with increasing observation time (left to right).

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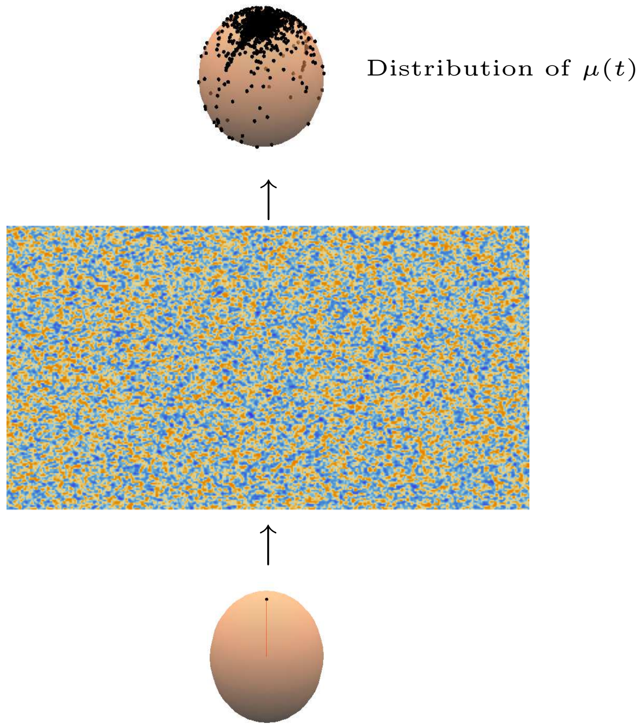
CPP formulation for multiple scattering



- $\mu(t) = \prod_{i=1}^{N(t)} X_i \mu_0$
- $mu(t)$ and μ_0 are $\mathcal{S}^2$-valued
- X_i represent the "random scatterers effect"
- X_i are $SO(3)$-valued
- The pdf of X_i is the *phase function*
- $N(t)$ is a Poisson process with parameter λ
- $\lambda = 1/\ell$, with ℓ: mean free path (normalized velocity)
- μ_0 is a Dirac at the north pole

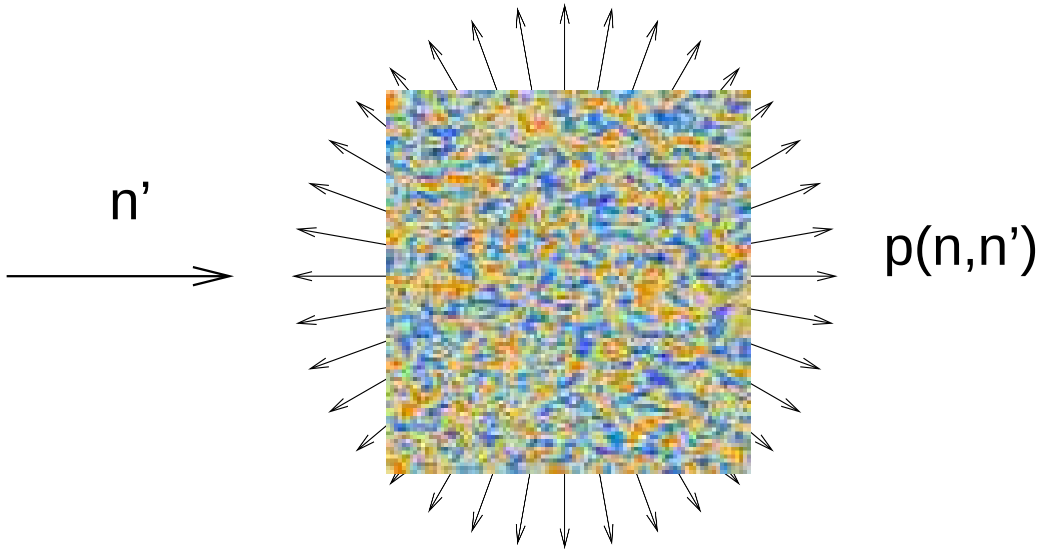
⋆ Decomposition of the solution into orders of scattering

$$I = N(0)I^{(0)} + N(1)I^{(1)} + \cdots + N(k)I^{(k)} + \cdots$$

$I^{(k)}$: Angular probability distribution of energy after *exactly* k scattering

$N(k)$: probability that the energy has been scattered *exactly* k times

⋆ Description of scattering anisotropy



$p(\mathbf{n}, \mathbf{n}')$: normalized phase function and $\int_{4\pi} p(\mathbf{n}, \mathbf{n}') d^2n' = 1$

⋆ Intensity distribution after k scattering events

Incoming plane wave

$$I^{(0)} = \delta(\mathbf{n} - \mathbf{n}')$$

After a single scattering event:

$$I^{1} = p(\mathbf{n}, \mathbf{n}')$$

Recurrence Formula:

$$I^{(k)} = \int_{4\pi} p(\mathbf{n}, \mathbf{n}') I^{(k-1)}(\mathbf{n}') d^2 n'$$

$\longrightarrow$ Repeated convolutions on the unit sphere

Simple case: $p(\mathbf{n}, \mathbf{n}') = f(\mathbf{n}.\mathbf{n}') = f(\cos\theta)$

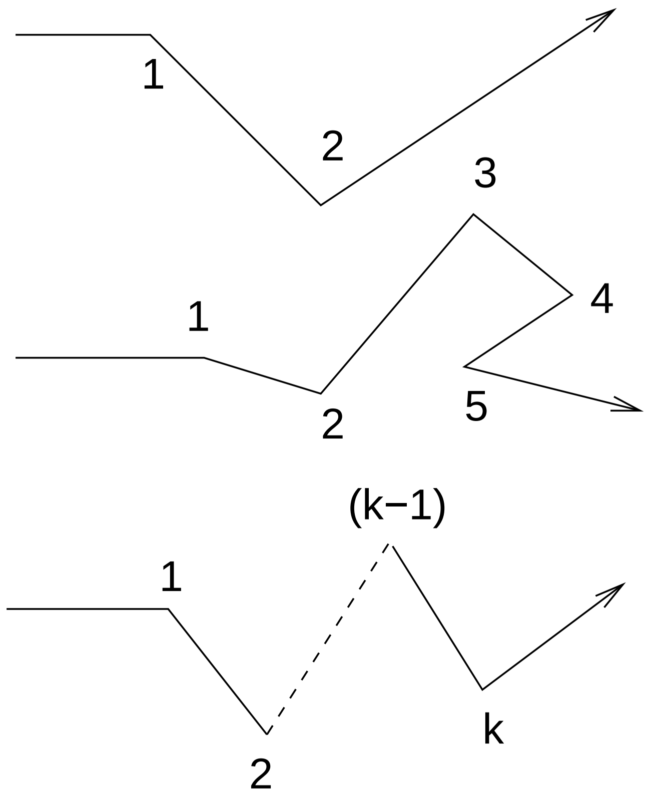
$$f(\cos\theta) = \sum_{l} \hat{f}^{l} P_{l}(\cos\theta)$$

Expansion in Legendre series:

$$I^{(k)}(\theta) = \sum_{\delta} (\hat{f}^{l})^{k} P_{\delta}(\cos\theta)$$

$f_1 = g$ **is the mean cosine of the scattering angle** (θ) **= anisotropy parameter**

⋆ Probability distribution of scattering events



Poisson Distribution

$$N(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$\lambda = \frac{t}{\tau}$$

t : **propagation time;** τ : **scattering mean free time**

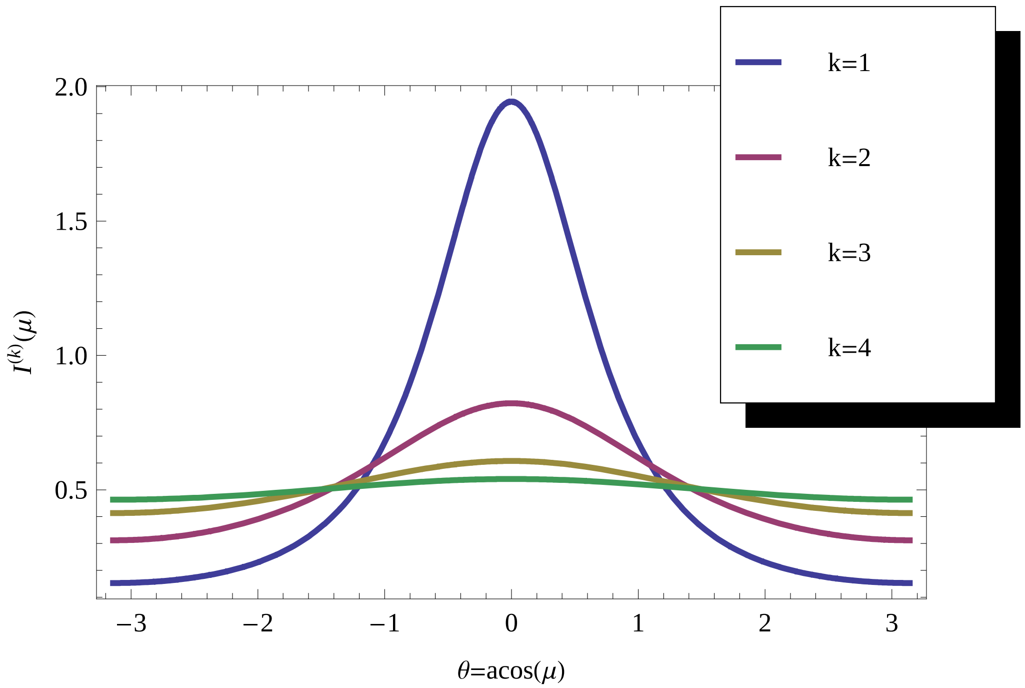
$\star$ A simple example: Henyey-Greenstein phase function

$$p(\cos\theta, g) = \frac{1-g^2}{2\left(1-2g\cos\theta+g^2\right)^{3/2}} \qquad \cos\theta = \mathbf{n}\cdot\mathbf{n}'$$

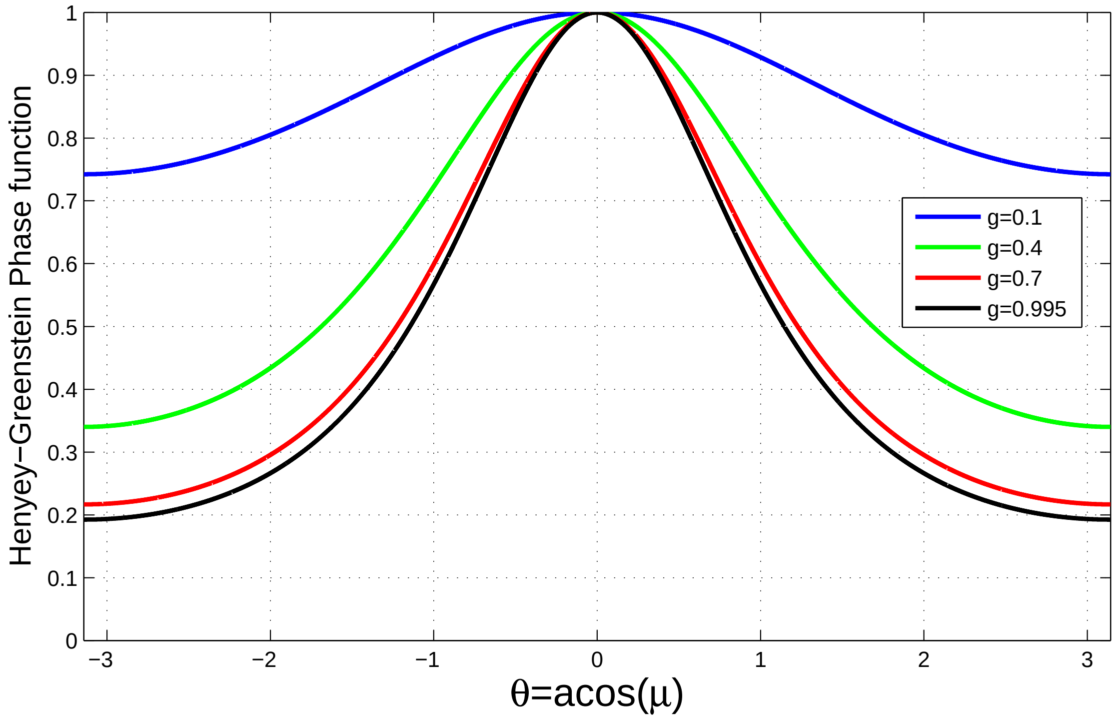
$g = \int_{-1}^{1} p(\cos\theta)\cos\theta d\cos\theta$ is the anisotropy parameter. After k scattering:

$I^{(k)} = p(\cos\theta, g^k)$

Example for $g = 0.4$



⋆ **Examples for different g**



Note: notation abuse: $\cos\theta = \mu$, the cosine of the scattering angle.

⋆ **The noncommutative harmonic analysis point of view**

The CPP model:

$$\mu(t) = \prod_{i=0}^{N(t)} X_i \mu_0$$

The probability density function of $\mu(t)$ is given by:

$$p_{\mu(t)} = \sum_{n=0}^{\infty} \left[p(x|N(t)=n) * p_{\mu_0})\right] p(N(t)=n) = \sum_{n=0}^{+\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \left(p_x^{\otimes n} * p_{\mu_0}\right)$$

Assumptions: μ_0 is at the north pole and p_x, the *phase function*, is **inverse invariant**

Then:

- $D^l_{mn}(\phi, \theta, \psi) \to P^l(\cos\theta)$
- $p_x(\cos\theta) = \sum_{l \geq 0}(2l+1)\hat{f}^l P^l(\cos\theta)$, with $\hat{f}^l = g^l$ in the Henyey-Greenstein case.

$$\Rightarrow \boxed{p_{\mu(t)} = \sum_{l \geq 0}(2l+1) \exp\left(\lambda t(\hat{f}^l - 1)\right) P^l(\cos\theta)}$$

⋆ The noncommutative harmonic analysis point of view

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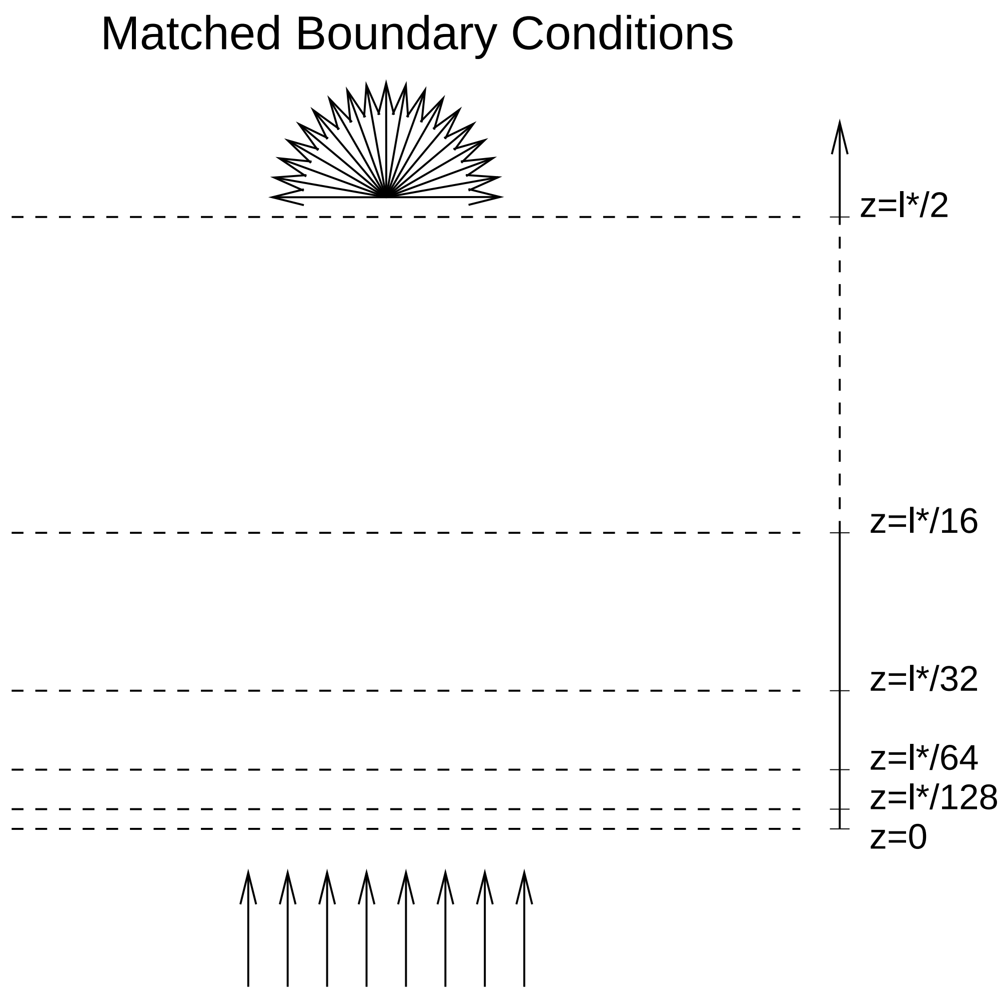
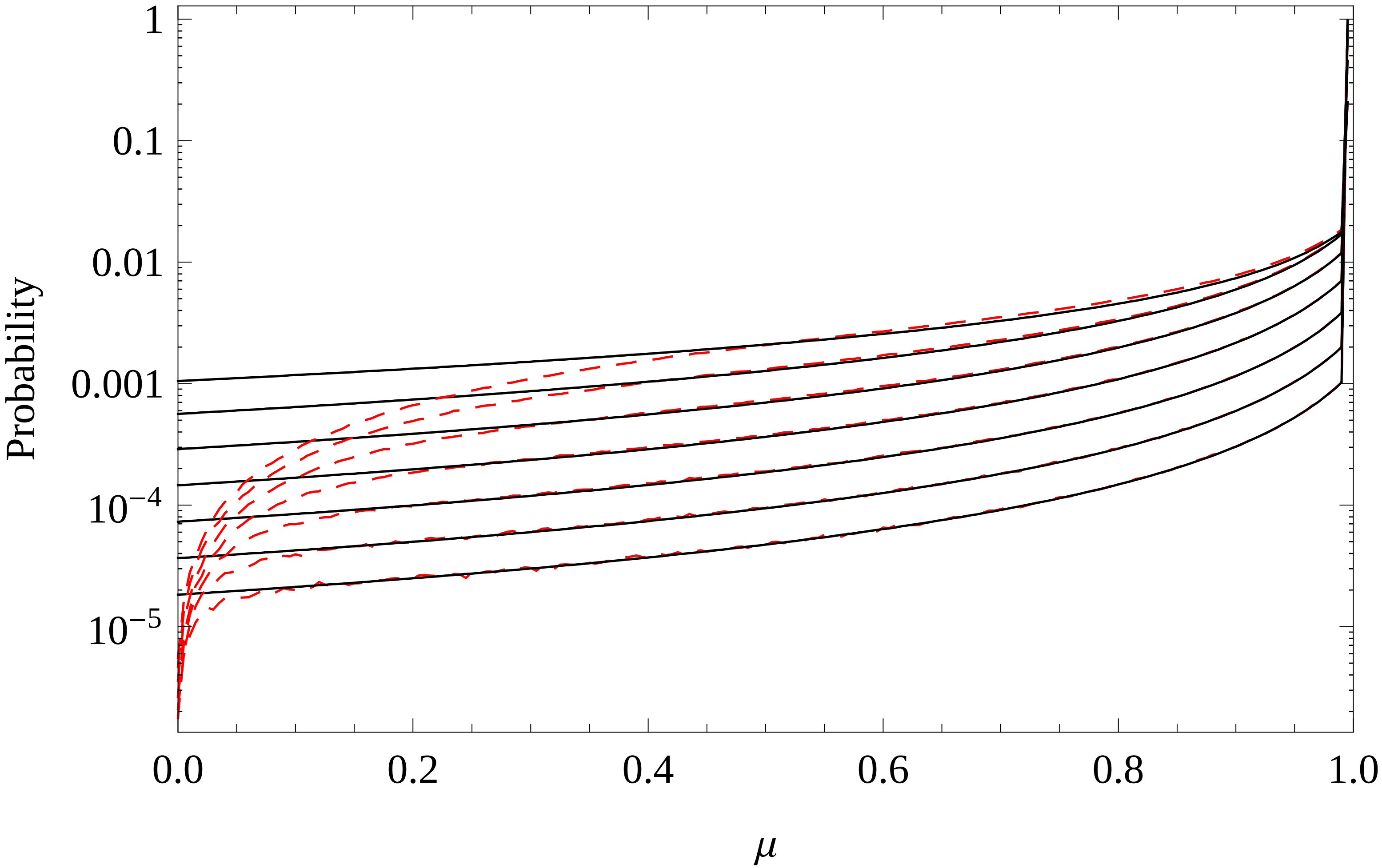
Simulation: a slab of random medium



z : **slab thickness** $\quad \ell^* = \ell/(1-g)$: **transport mean free path**

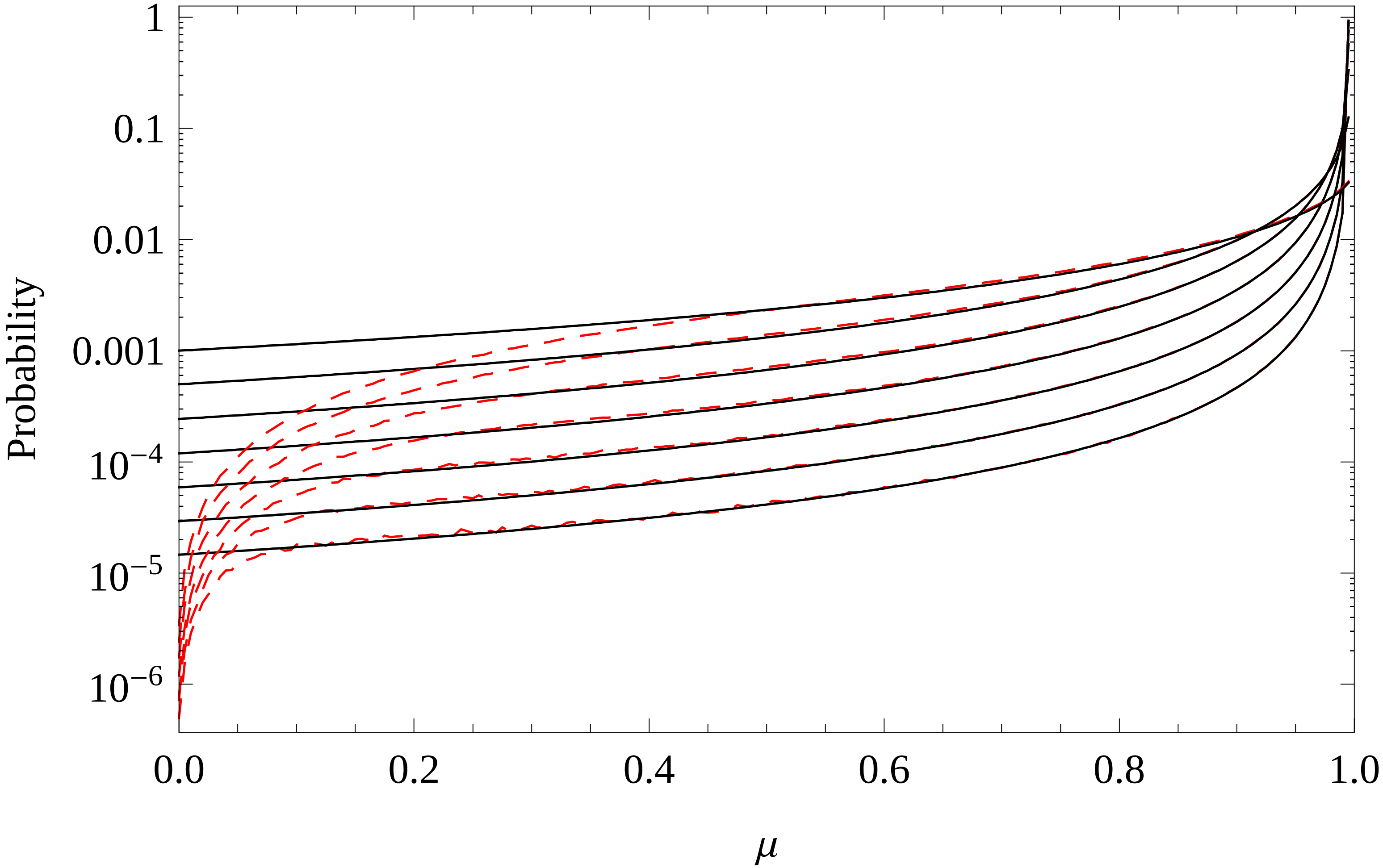
ℓ : **mean free path** $\quad$ **Poisson distribution:** $\lambda = \frac{1}{\ell}$

Henyey-Greenstein Weak Anisotropy $g = 0.7$



Comparison between simple **CPP Analytical Formula** and **Monte-Carlo simulations**

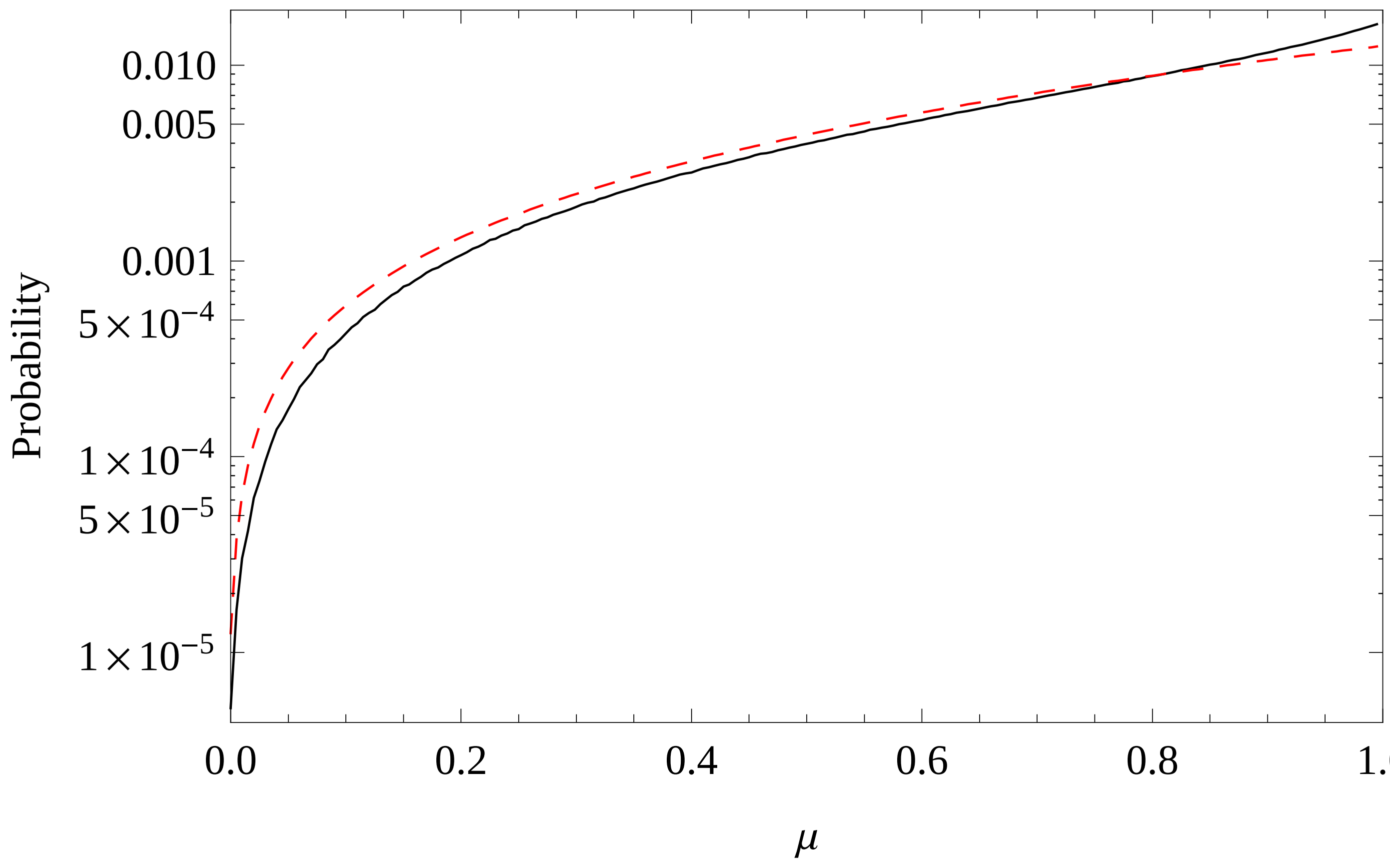
Henyey-Greenstein Large Anisotropy $g = 0.95$



Non-uniform approximation

Remark: Much faster than M-C Simulations for large anisotropy

Henyey-Greenstein Intensity Distribution for z=l^*, $g = 0.95$



Comparison of **Monte Carlo simulation** with **Diffusion approximation:**

$$I(\mu) = \mu\left(1 + \frac{3\mu}{2}\right)$$

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Decompounding: a nonparametric estimation problem

⋆ Assumptions:

- we are given some **noise free** realizations of $\mu(t)$ at time T ($\Leftrightarrow$ depth z)
- $\mu(T)$ is modeled as a CPP and $p_{\mu(T)} = \sum_{l \geq 0} (2l+1) \hat{\mu}^l P^l(\cos\theta)$
- The dataset is: $[\mu_1, \mu_2 \ldots \mu_N]$ (cosine of scattering angles)
- τ is supposed known $\Leftrightarrow \ell$ is known.

⋆ Empirical estimator of $\hat{\mu}^l$:

$$\widetilde{\hat{\mu}^l} = \frac{1}{N} \sum_{n=1}^{N} P^l(\mu_n)$$

This is an **unbiaised** estimator with **variance**: $N^{-1}(\mathbb{E}[(P^l)^2(\mu)] - (\hat{\mu}^l)^2)$

Estimator derived from the fact that: $\hat{\mu}^l = \int_{-1}^{1} p(\mu(T)) P^l(\cos\theta) d\cos\theta$

Phase function and Anisotropy estimation

- **Phase function estimate**

Using the coefficients $\widetilde{\hat{\mu}}^l$, it is possible, by inversion of the characteristic function, to estimate the Legendre coefficients of $p(\cos\theta, g)$, the phase function, with:

$$\widetilde{\hat{f}}^l = \frac{\tau}{T} \ln \widetilde{\hat{\mu}}^l + 1$$

Then, the phase function can be reconstructed:

$$\widetilde{p}(\cos\theta, g) = \sum_{l=0}^{L_{Max}} (2l+1) \widetilde{\hat{f}}^l P^l(\cos\theta)$$

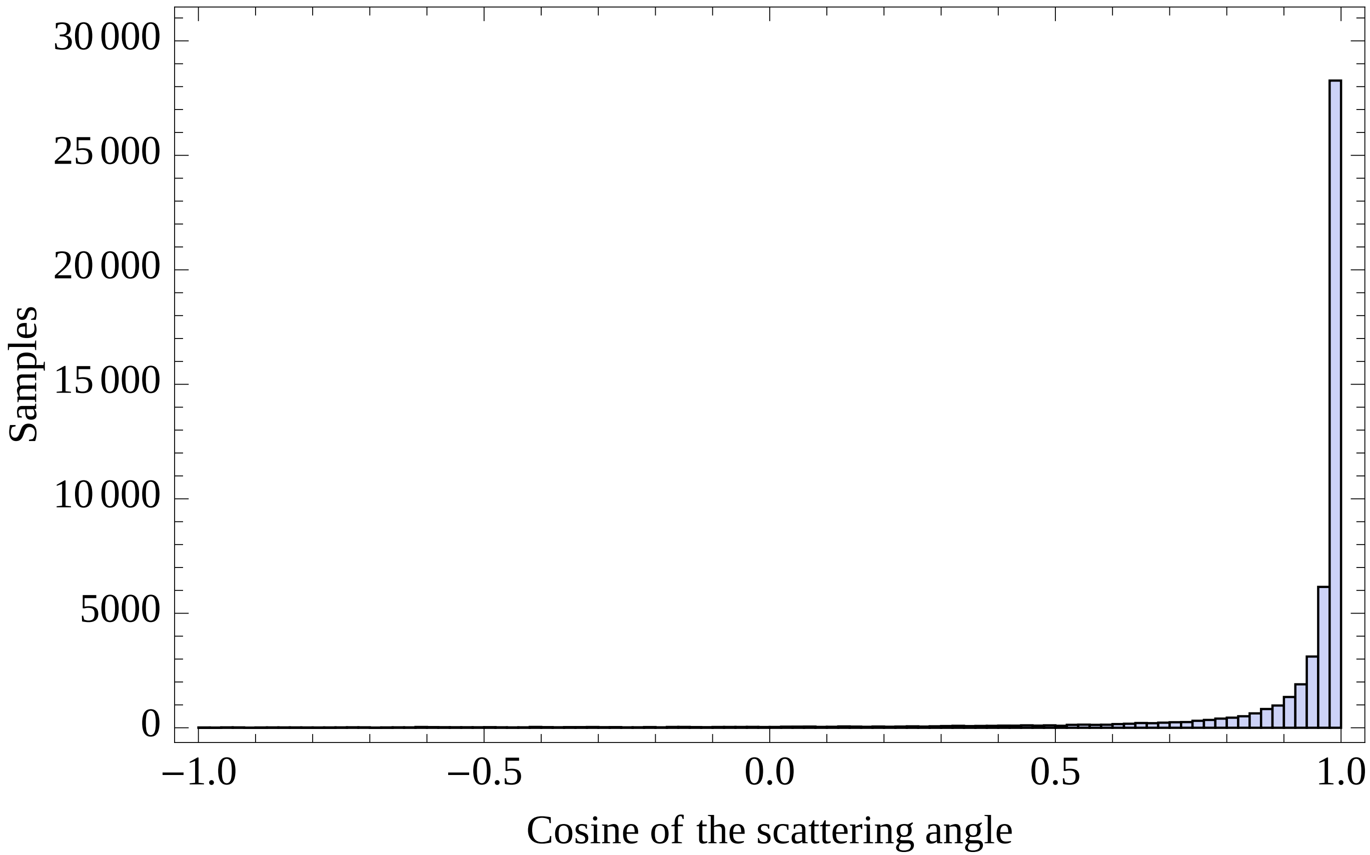
- **Estimate for g**

The fact that p is Henyey-Greenstein (Legendre coefficients of the form g^l) allows to give an estimator for the anisotropy g:

$$\widetilde{g} = \left(\frac{\tau}{T} \ln \widetilde{\hat{\mu}}^l + 1 \right)^{1/l}$$

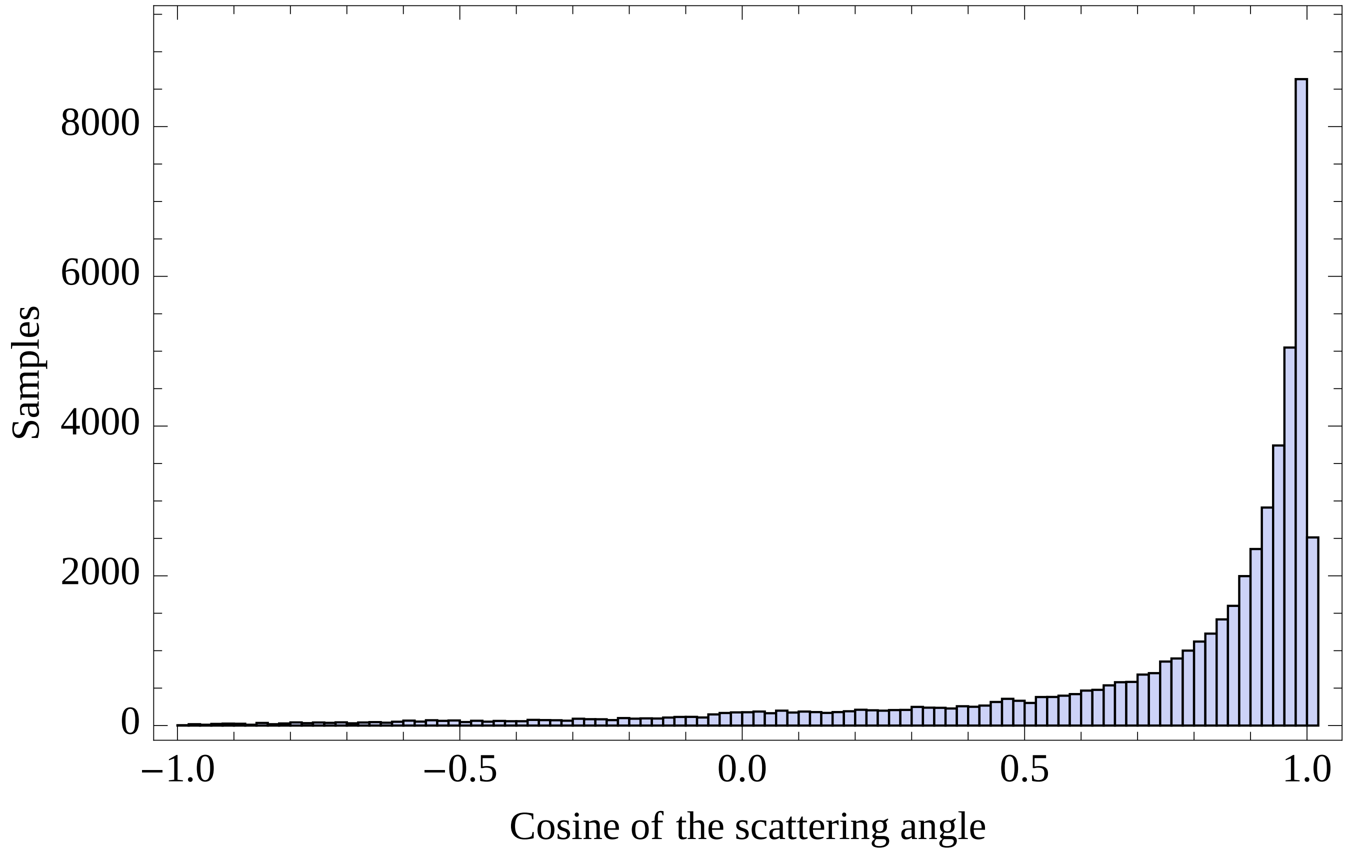
Recall that this estimator needs the knowledge of τ.

Decompounding: simulations



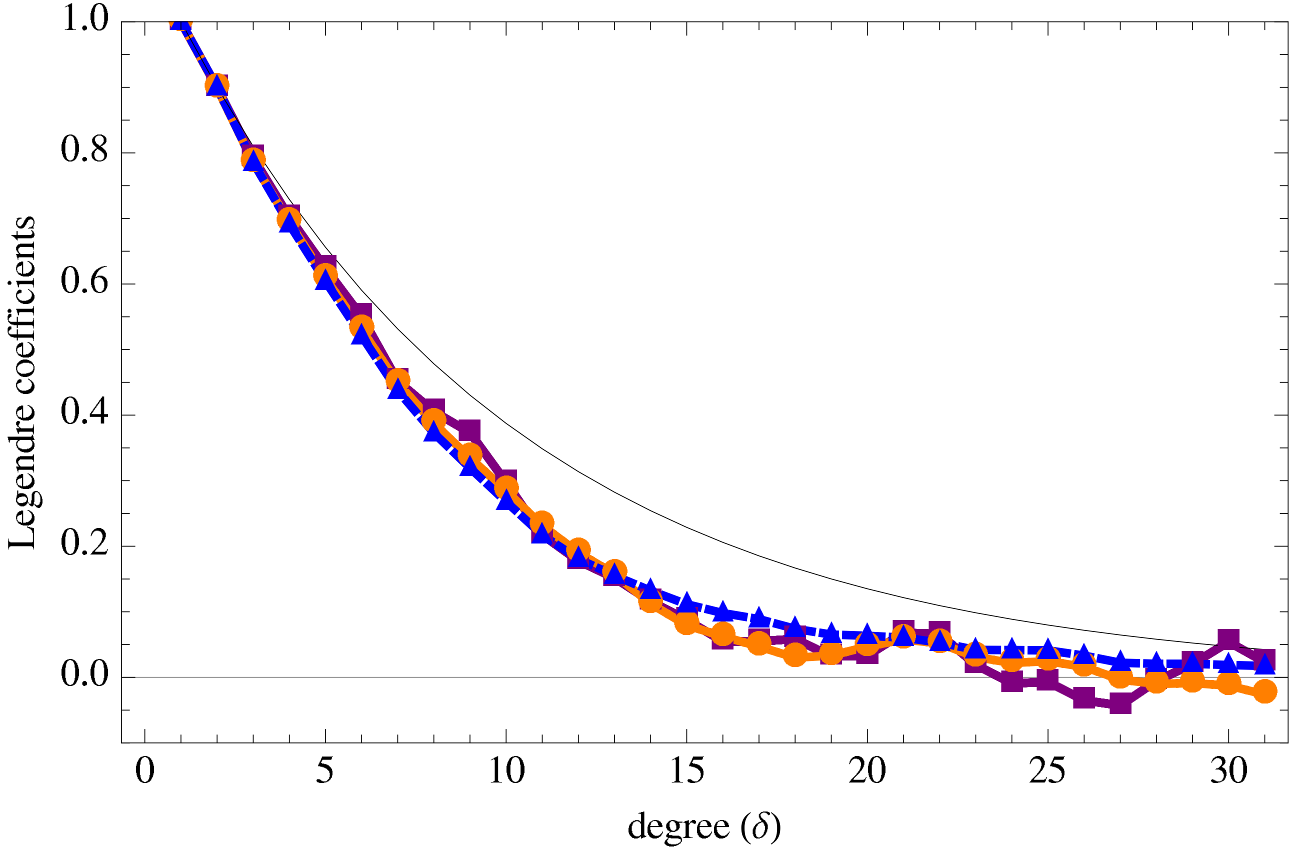
Phase function used in the CPP simulation. $g = 0.9$.

Decompounding: CPP distribution



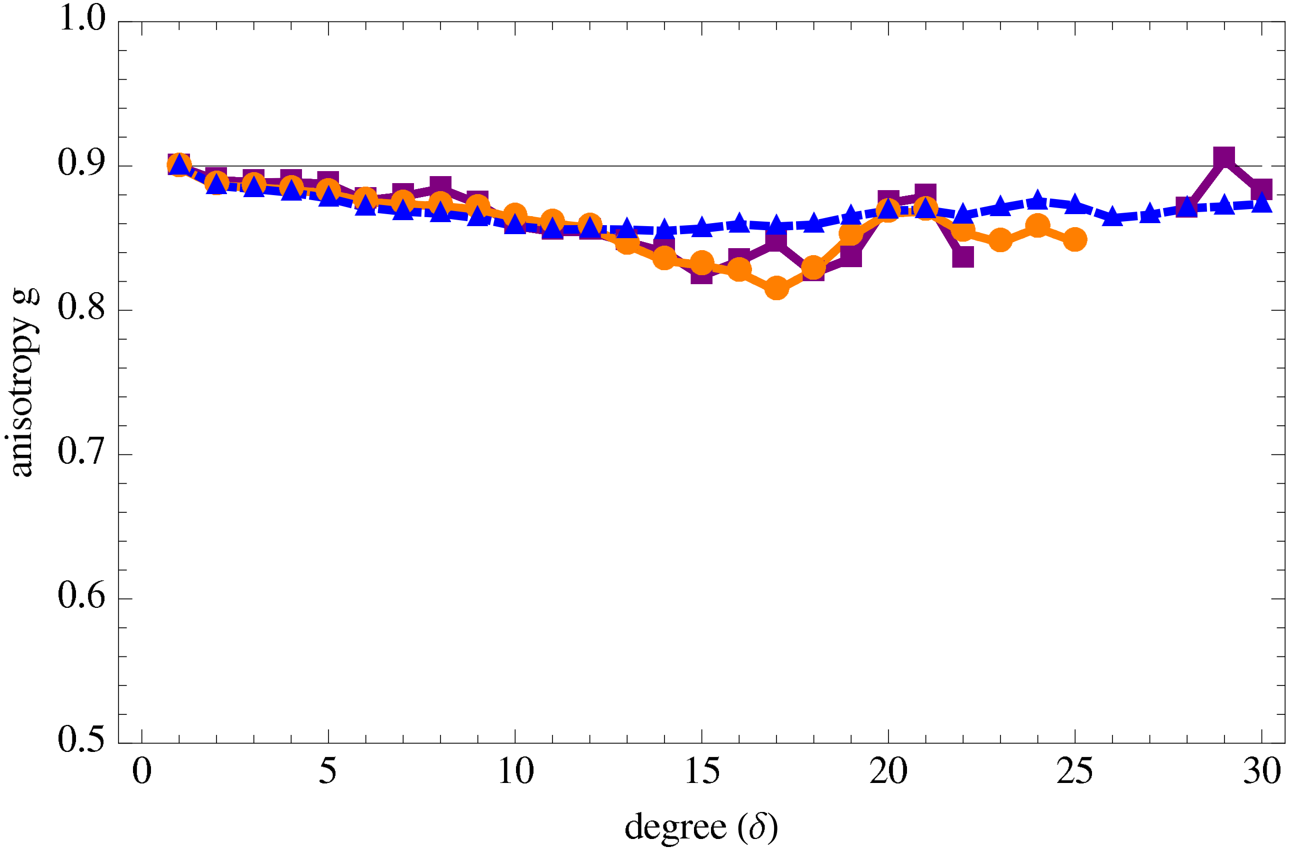
Scattering angles distribution ($p_{\mu(T)}$) with a CPP where $\lambda T = 4$ (average number of scattering events in time T).

Decompounding: phase function estimation



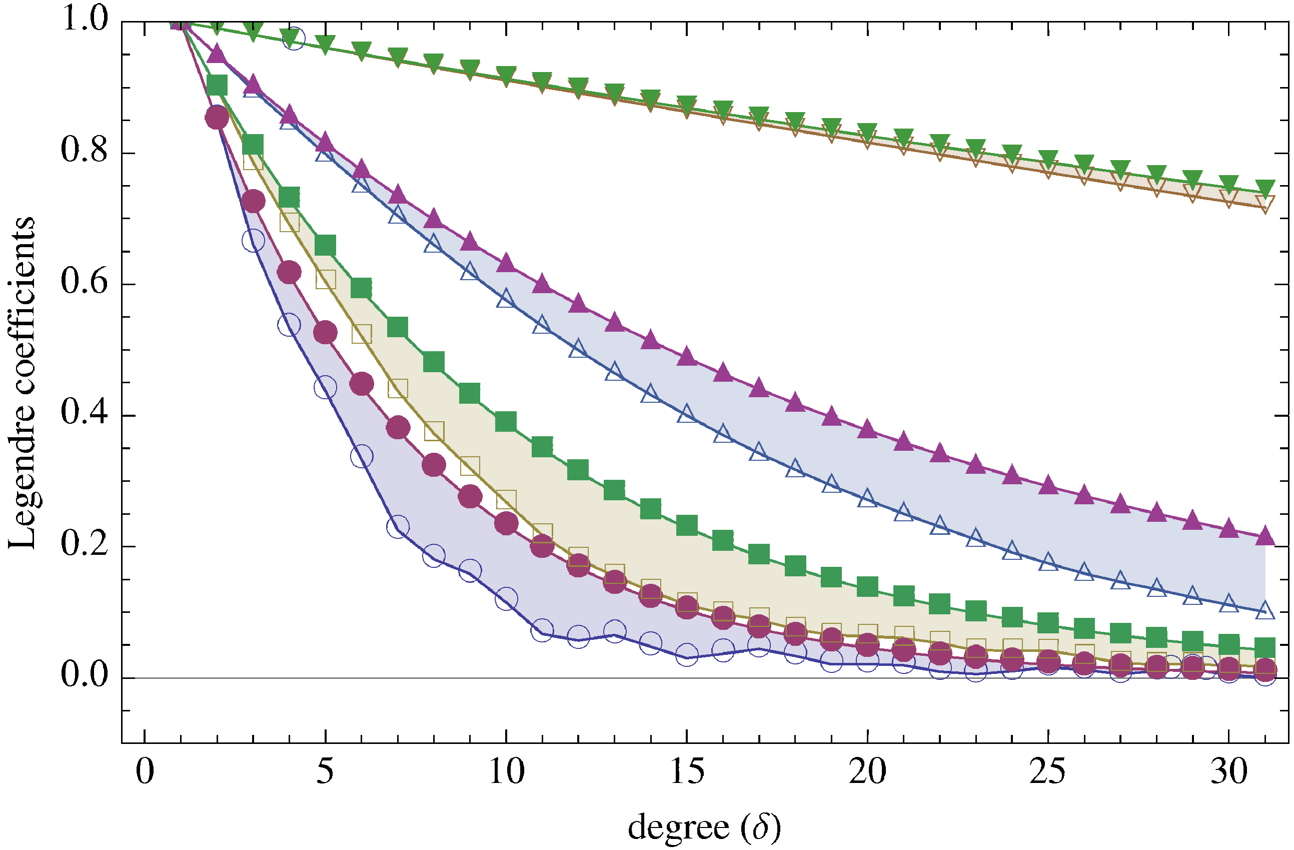
Decompounding: Estimated Legendre coefficients of the H-G phase function with $N = 500$ (purple), $N = 5000$ (yellow) and $N = 50000$ (blue) samples.

Decompounding: anisotropy estimation



Decompounding: Estimated parameter g of the H-G phase function with $N = 500$ (violet), $N = 5000$ (yellow) and $N = 50000$ (blue) samples.

Decompounding: influence of g



Decompounding: Error on Legendre coefficients of the H-G phase function for $g = 0.85$ ($\circ$), $g = 0.9$ ($\square$), $g = 0.95$ ($\triangle$) and $g = 0.99$ (∇)

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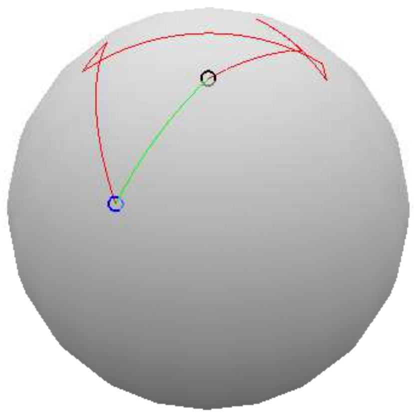
CPP and geometric phase

⋆ **Model**

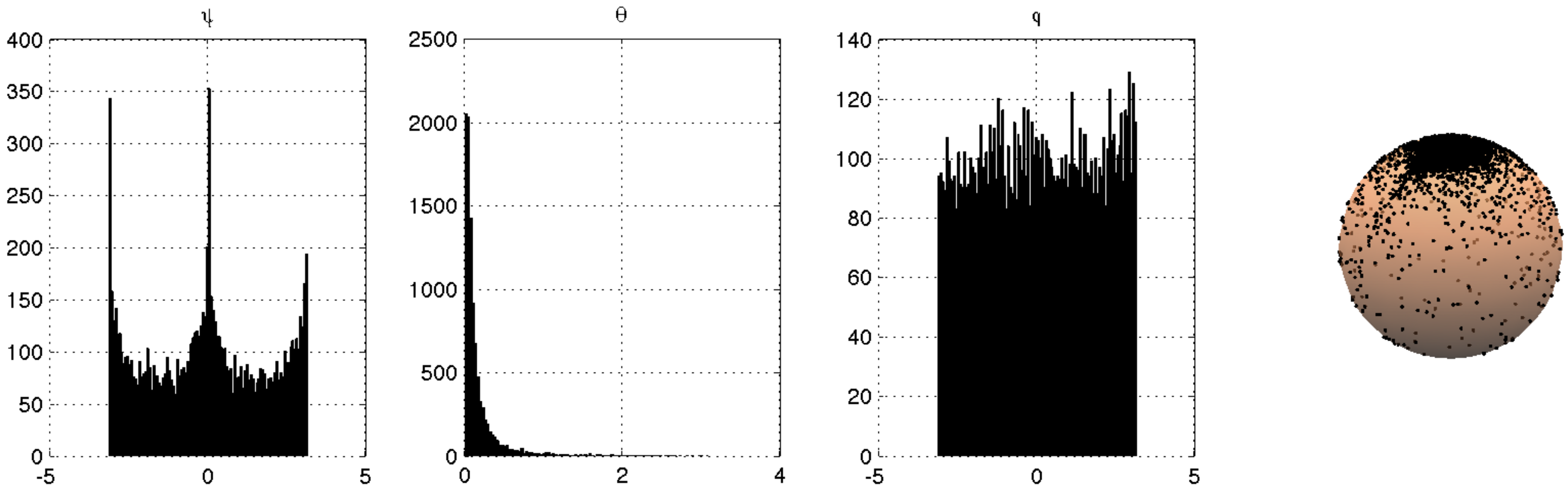
- Polarization & CPP: Parallel transport of the polarization plane over $\mathcal{S}^2$.
- Polarized CPP $Leftrightarrow$ CPP on $SO(3)$
- Geometric phase → influence on third parameter distribution
- Brownian motion on $\mathcal{S}^2$: probability of solid angle (path integrals).
- Polarized CPP: angle distribution ?
- Practical issue: observable ?

CPP and geometric phase



Area (closed by geodesic in green) $\propto$ geometric phase

CPP and geometric phase



Euler angle distribution for a CPP on $SO(3)$

Geometric phase information in ψ ?

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Conclusions

- Random processes and multiple scattering
- CPP allows modelization of forward multiple scattering
- Decompounding: estimation of phase function, with ℓ known
- Inference on heterogeneous media
- Small angle approximation
- Parametric estimation
- Extension to include spatial information
- Polarization: CPP on $SO(3)$ and geometric phase