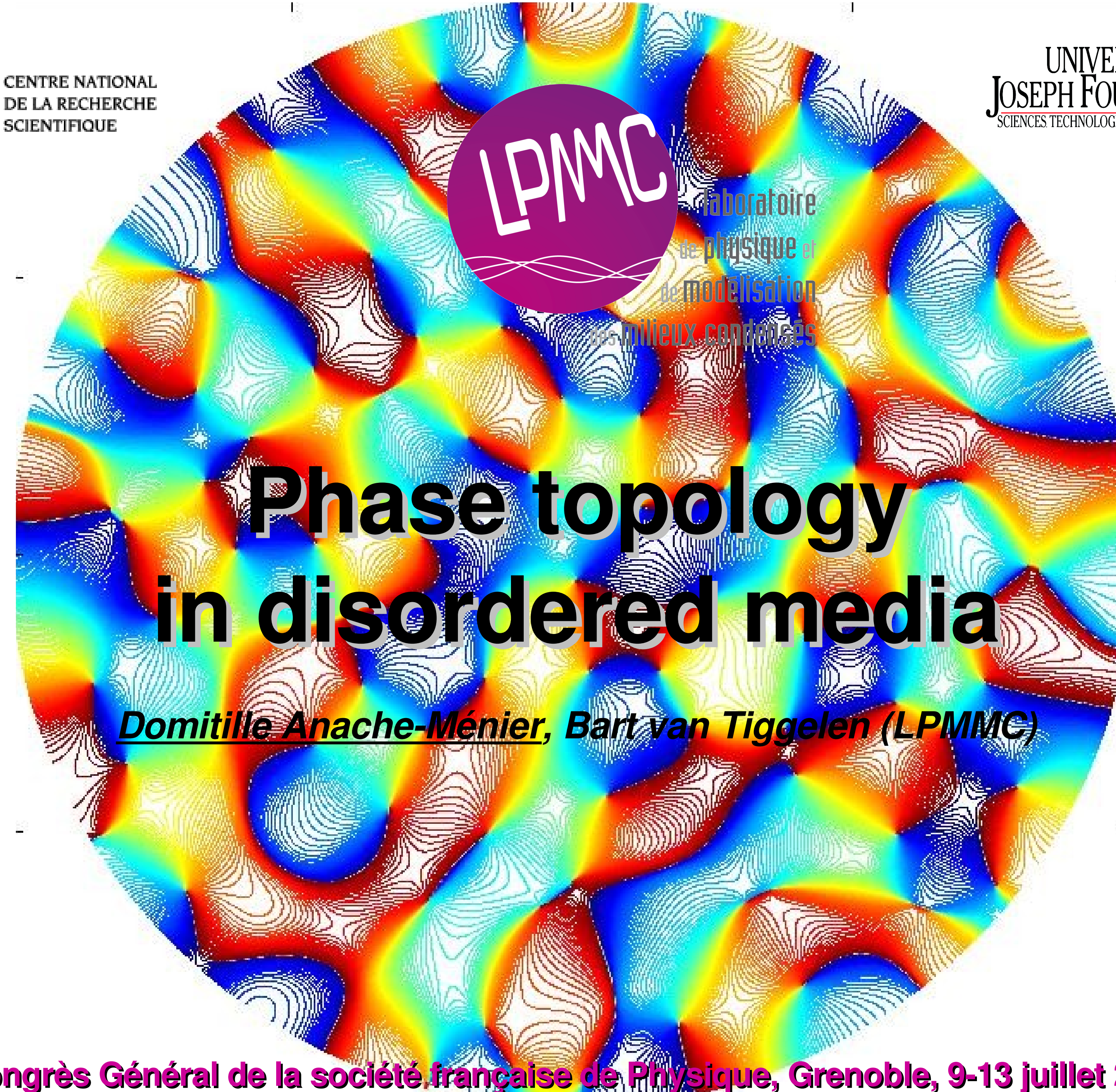
CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

LPMMC laboratoire de physique et de modélisation des milieux condensés

UNIVERSITE JOSEPH FOURIER GRENOBLE 1
SCIENCES TECHNOLOGIE MEDECINE



Phase topology in disordered media

Domitille Anache-Ménier, Bart van Tiggelen (LPMMC)

Congrès Général de la société française de Physique, Grenoble, 9-13 juillet 2007

Outline

Motivation: why investigate phase in multiply scattering media?

- It is a genuine property of waves
- There have been few studies of phase compared to amplitude or intensity.
- In microwaves, acoustics and seismology phase can be measured directly

A superposition of waves scattered by a disordered medium gives rise to a speckle pattern which presents a complicated network of phase vorticies

Layout:

Phase Topology:

topological charge Q

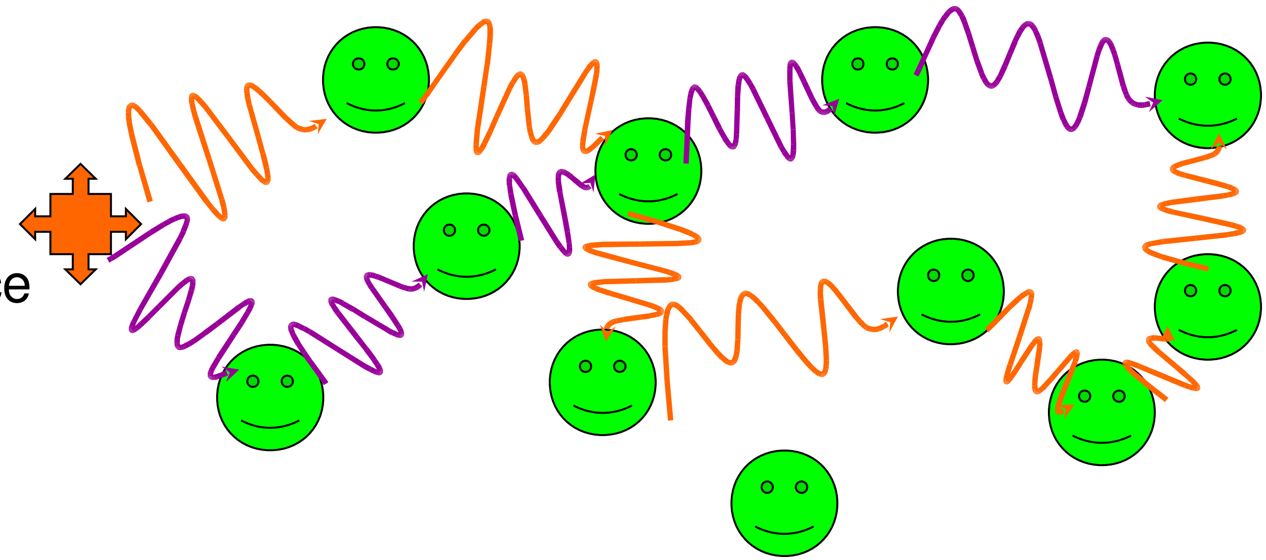
fluctuations of Q in space

fluctuations of Q in a wave guide

Result: *role of mean free path and of the boundaries*

Field in disordered media

Complex scalar field: $\Psi = A\exp(i\Phi) = \eta + i\xi$



Sum of partial waves: $\Psi = A\exp(i\Phi) = \sum_{\alpha} A_{\alpha}\exp(i\Phi_{\alpha})$

After a few mean free path phase becomes random and partial waves become independent so aplying central limit theorem:

$$P(\Psi_1,\ldots,\Psi_N) = \frac{1}{\pi^N det(C)}\exp\left(-[\Psi_1^*\ldots\Psi_N^*]\,C^{-1}\begin{bmatrix}\Psi_1\\ \vdots \\ \Psi_N\end{bmatrix}\right)$$

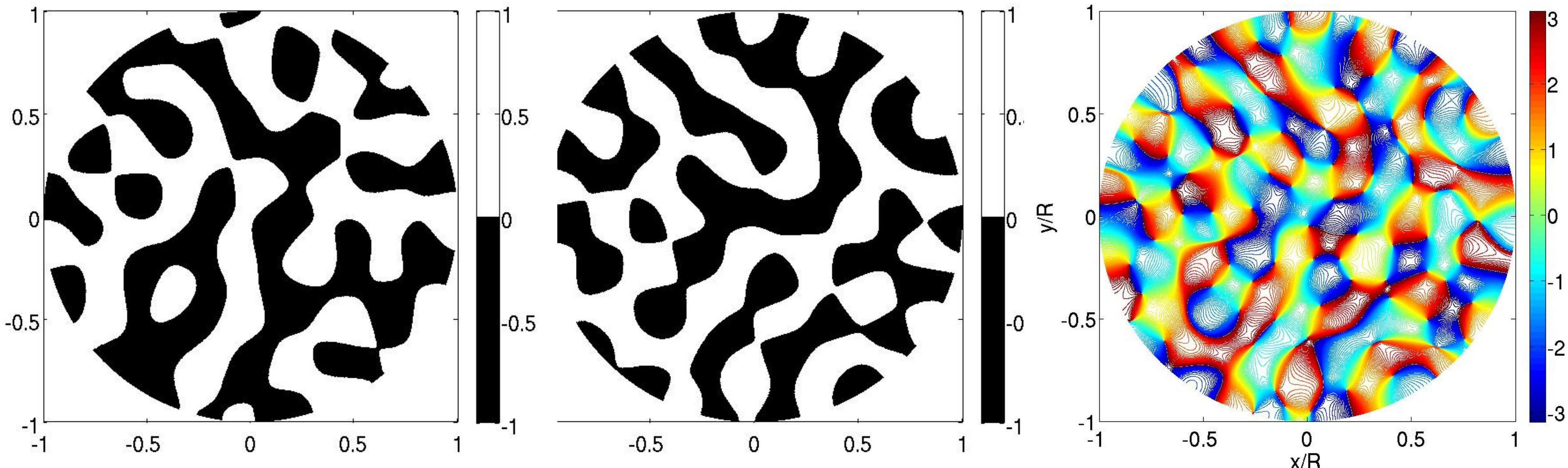
Phase singularity

PHASE SINGULARITY:

Recall: $\Psi = A\exp(i\Phi) = \eta + i\xi$

When the field cancels η=0 and ξ=0 then A=0 but the phase Φ is left undefined

EXAMPLE:

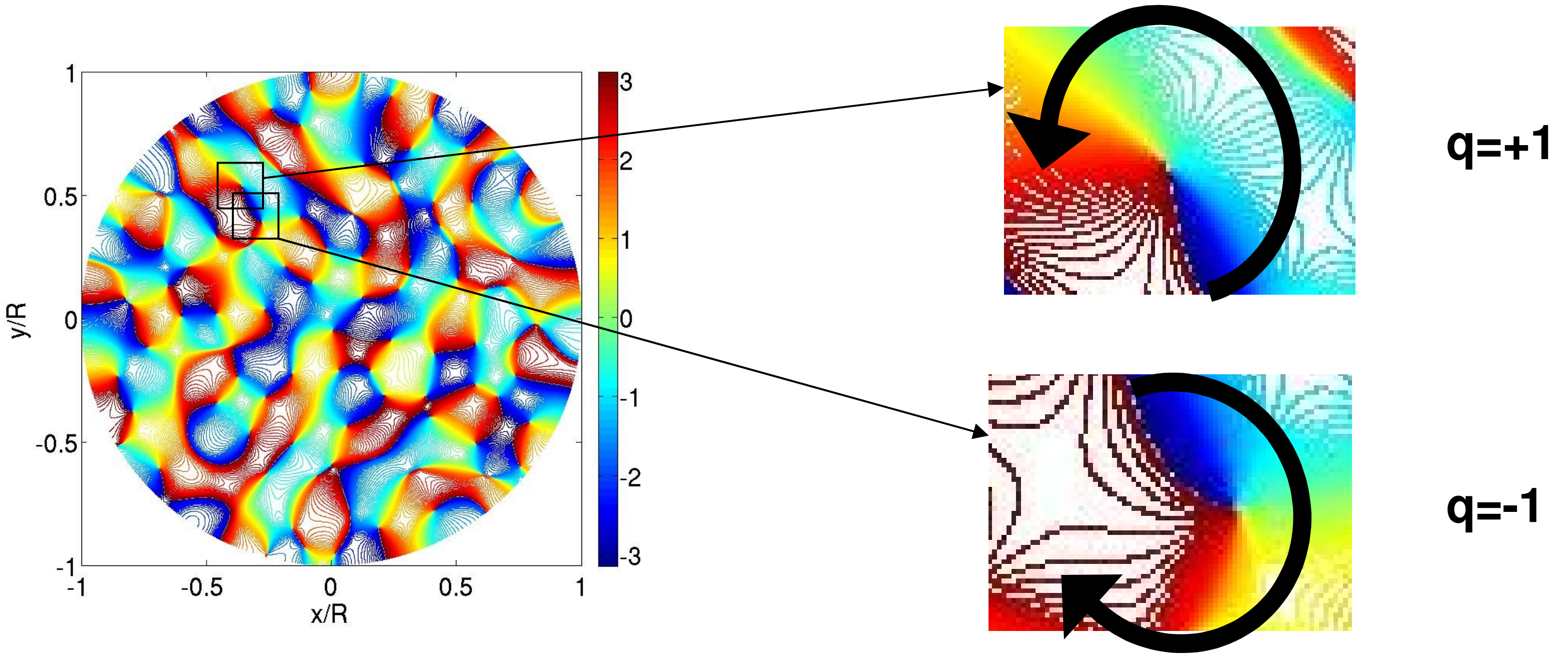


Re(Ψ) Im(Ψ) equiphases

⇒Singularities are points on a surface and locate at the intersection of the equiphases

Topological charge

TOPOLOGICAL CHARGE OF A SINGULARITY = sign of the phase vortex



TOTAL TOPOLOGICAL CHARGE PRESENT ON THE SURFACE:

$$Q = \sum_i q_i$$

Stokes' theorem: $\oint_\Gamma d\mathbf{r} \cdot \nabla\Phi(\mathbf{r}) = 2\pi Q$

Gaussian speckle

Exemple: monochromatic waves in space (Berry 78):

For a plane section, dislocation point density: $d = \frac{k^2}{6\pi} = \frac{2\pi}{3\lambda^2}$

Surface of a speckle spot: $\sim\lambda^2$

⇒ About 2 singularities for each speckle spot

Statistics of topological charge:

As we average over the disorder:

$\langle Q \rangle = 0$

$\langle Q^2 \rangle \neq 0$

Motivation: study the dependence of $\langle Q^2(R) \rangle$ with the mean free path and the surface size in space and in a wave guide.

Charge screening

Independent charges: $$< Q^2(R) >=< (\sum_{n=1}^{N} q_n)^2 >= N = \pi R^2 d$$

=> Fluctuations proportional to the surface

SCREENING EFFECT (~electrical charge in ionic fluids or plasma)

Sign principle The continuity of the field imposes that two connected singularities are of **opposite sign.**

Freund et al.

Wilkinson & Freund (98) study a random superposition of plane waves in space:

⇒ Fluctuations $<Q^2(R)>$ proportional to the radius R

Berry & Dennis (2000) use gaussian smoothed boundaries:

=> Fluctuations $<Q^2(R)>$ independent of the number of singularities and independent of the surface

Calculation method

Field correlation function $C = \langle \Psi(0)\Psi(x) \rangle$

3D:

$$C(x) = \frac{\sin(kx)}{kx}\exp\left(-\frac{x}{2\ell}\right)$$

$$C_{\Phi'} = \frac{1}{2}(logC)'' log(1 - C^2)$$

3D:

$$C_{\Phi'}(x > \lambda) \rightarrow \frac{1}{2x^2}\exp\left(-\frac{x}{\ell}\right)$$

Phase derivative correlation function

$$C_{\Phi'} = \left\langle \frac{\partial \Phi}{\partial \theta}(\theta)\frac{\partial \Phi}{\partial \theta}(\theta') \right\rangle$$

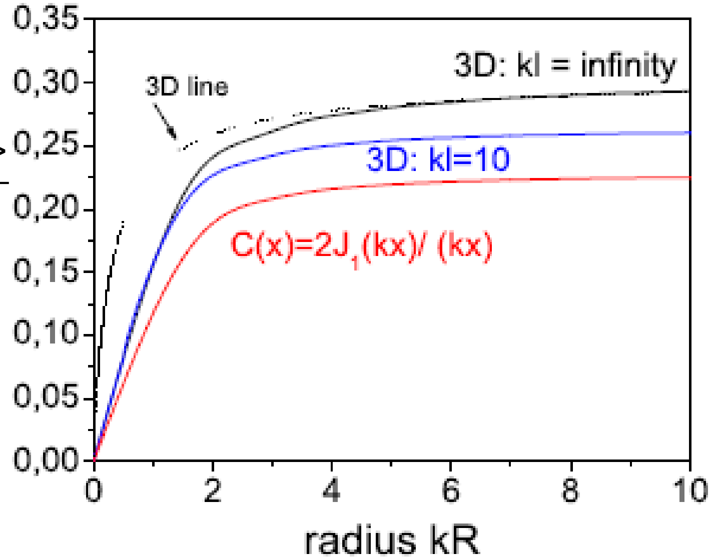
$$\langle Q^2(R) \rangle = \frac{1}{(2\pi)^2} \oint_{\Gamma(R)} \oint_{\Gamma(R)} d\mathbf{s} \cdot \langle \nabla \Phi(\mathbf{s}) \nabla \Phi(\mathbf{s}') \rangle \cdot d\mathbf{s}' = \frac{1}{2\pi} \int_0^{2\pi} d\Delta\theta \, C_{\Phi'}(\Delta\theta, R)$$

Topological charge variance $\langle Q^2(R) \rangle$ included in a circular surface S(R) with a contour Γ(R).

Infinite media: diffuse behaviour

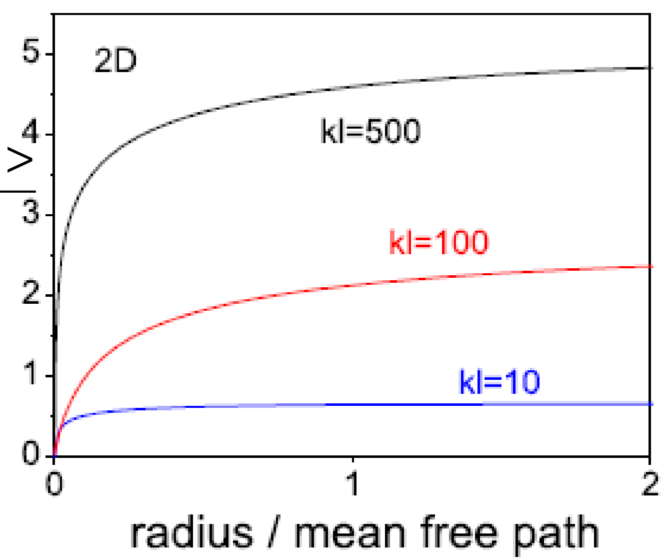
(a) 3D

$$\frac{< Q^2(R) >}{kR}$$



(b) 2D

$$\frac{< Q^2(R) >}{kR}$$



- Diffuse behaviour as found by Wilkinson and Freund
- 3D: very weak dependence on the mean free path ℓ
- 2D: <Q²> depends logarithmicaly on $k\ell$

EPL 2006

Wave guide (WG)

Motivation:

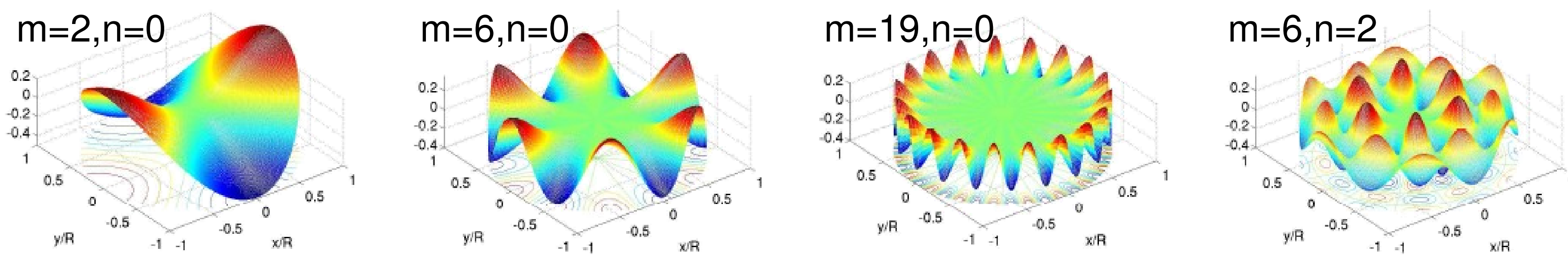
- configuration used in experiments (Sebbah, Genack et al.)
- Field is confined inside the WG => sure not to forget any contribution to the screening process.
- Study the influence of boundaries on the screening

System: a hollow conductive **cylindrical WG** with infinite length containing disorder.

Modes:

$$\Psi_{mnk}(\rho,\theta,z) \propto J_m\left(\frac{\alpha_{mn}\rho}{R}\right)\exp(im\theta)\exp(ikz) \quad \text{with} \quad J'_m(\alpha_{mn})=0$$

$$m \in \mathbb{Z},\ n \in \mathbb{N}$$



Dispersion relation: $\omega^2 = k^2c^2 + \alpha_{mn}^2 c^2/R^2$

Cut-off frequency: $\omega_{mn} = \alpha_{mn} c/R$

Calculation method

Green function:

$$G^{\pm}(\rho,\theta,z;\rho',\theta',z';\omega)=\sum_{m,n}\int dk\,\frac{\Psi_{mnk}(\rho,\theta,z)\Psi^{*}_{mnk}(\rho',\theta',z')}{\omega_{mn}(k)^2-\omega^2\pm\mathrm{i}\epsilon+\mathrm{i}\omega/\tau_{mn}}$$

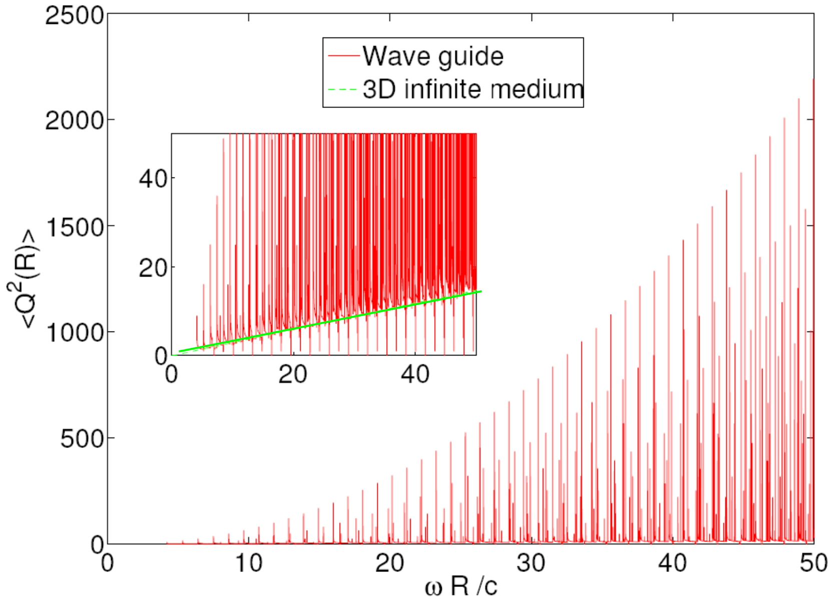
Field correlation function:

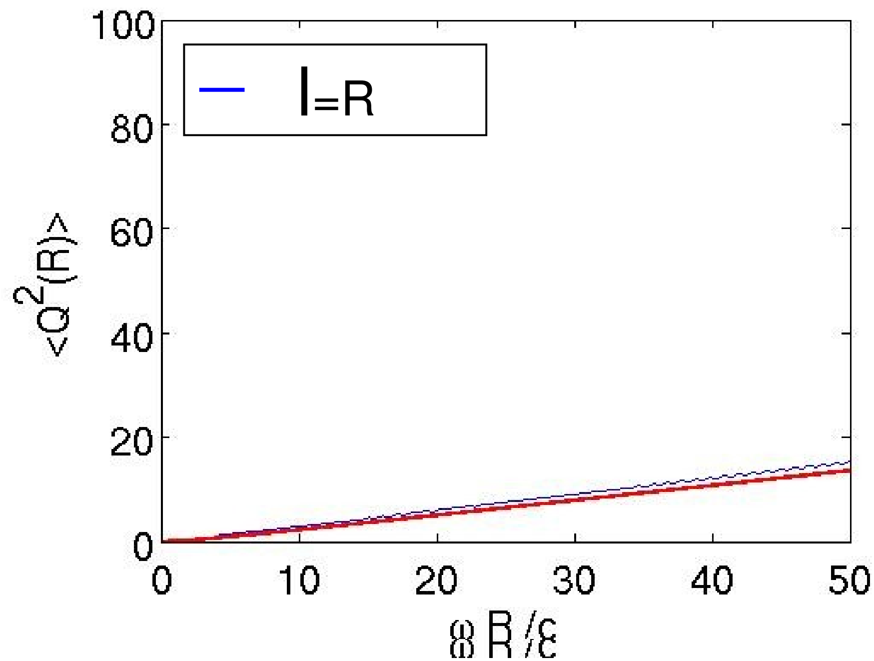
$$\begin{aligned}C_{\Psi}(\rho,\Delta\theta,\omega) &=<\Psi(\rho,\theta,z)\Psi(\rho,\theta',z)>\propto \mathrm{Im}\,G(\rho,\theta,z;\rho,\theta',z;\omega)\\ &\propto\sum_{m,n}\frac{1}{R}\cos(m\Delta\theta)\frac{2\alpha_{mn}^2}{\alpha_{mn}^2-m^2}\left|\frac{J_m(\frac{\alpha_{mn}\rho}{R})}{J_m(\alpha_{mn})}\right|^2 A_{mn}(\omega)\end{aligned}$$

Huge topological fluctuations

<Q²(R)> in an entire cross-section of a wave-guide:

Very large l :

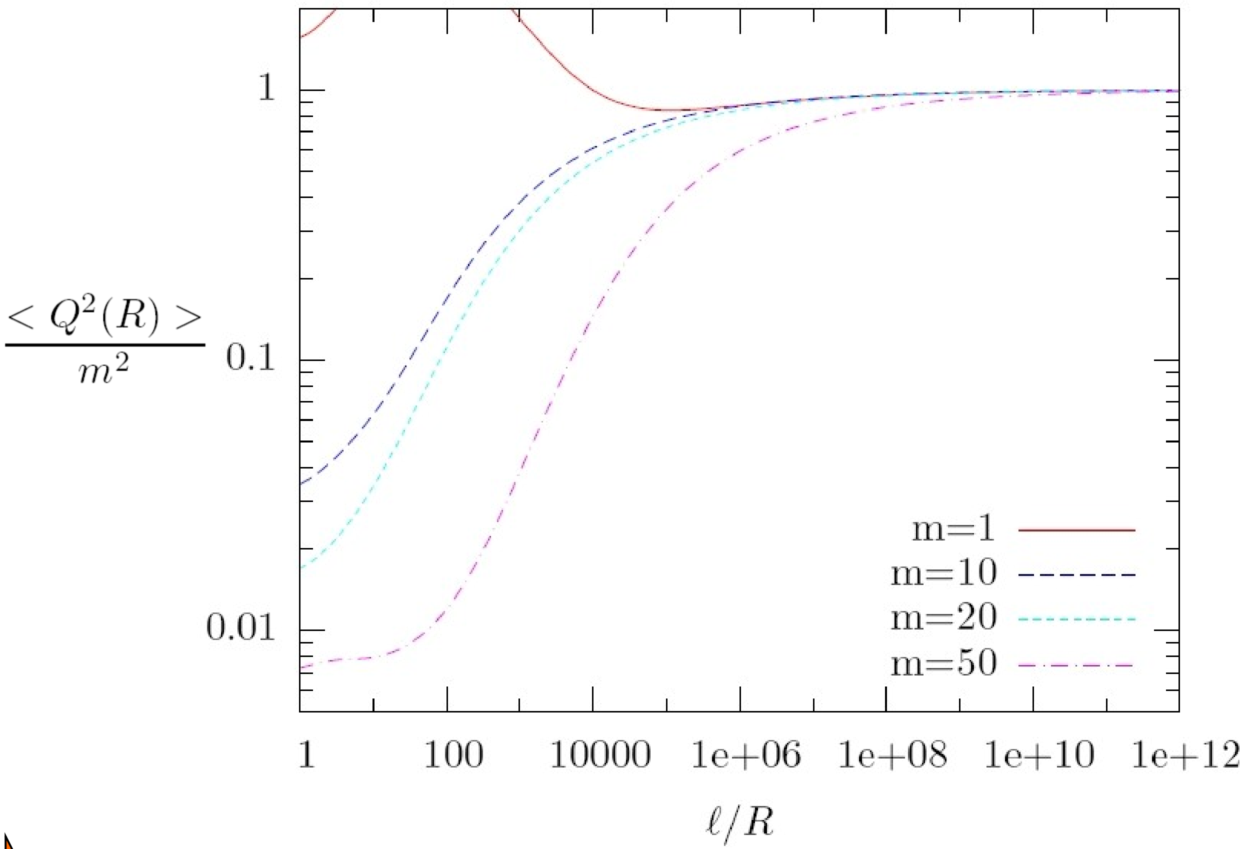




- Global behaviour consistent with the calculation for 3D infinite media
- Huge fluctuations at the cut-off frequencies of the wave guide
- <Q²(R)>=m^2 for l sufficiently large

Role of mean free path

Peak value of <Q²(R)> at different cutt-off frequencies ω_{mo}



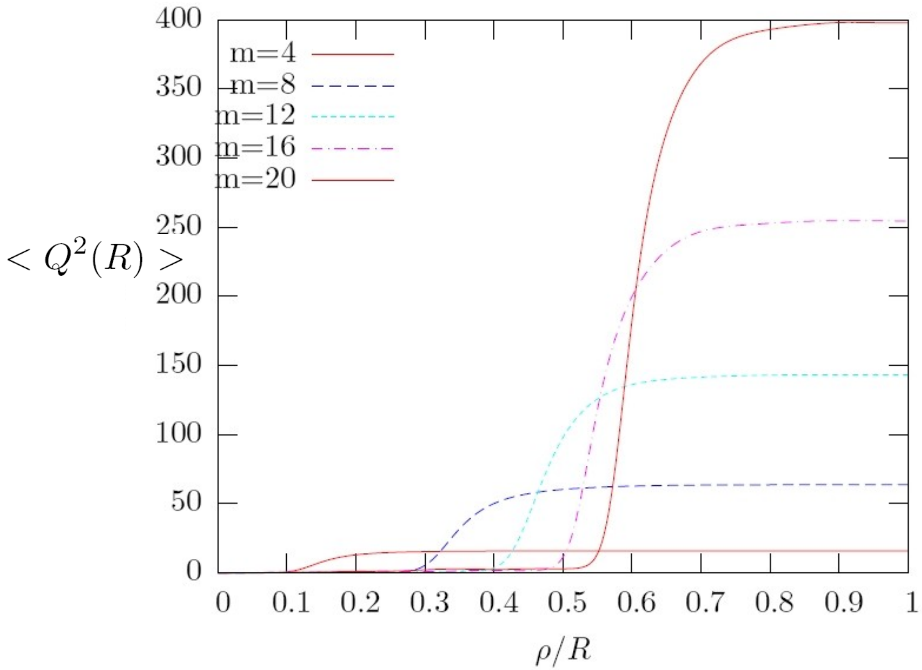
Slow dependence on the mean free path ℓ

No peaks for $l < R$

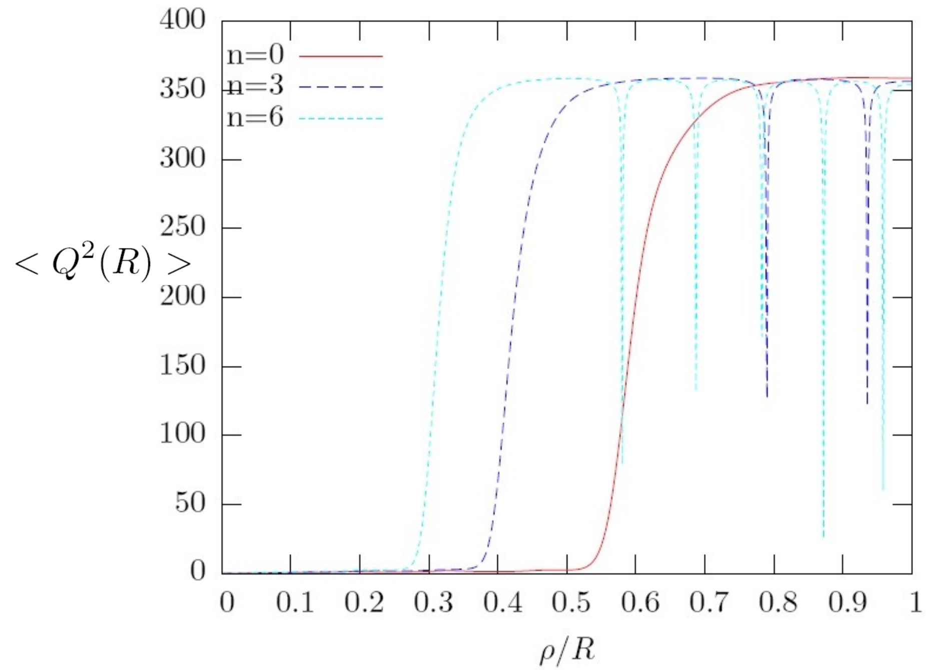
An effect near the edges

Peak value of $\langle Q^2(\rho) \rangle$ at different cutt-off frequencies ω_{mo}, $\rho<R$

(a) n=0



(b) m=19



$\rho<\rho_c$: screening (like in infinite media)

$\rho>\rho_c$: unscreened charges generating the huge $\langle Q^2(R) \rangle$ value

Simulation

Simulation of a random circular Gaussian complex wave field

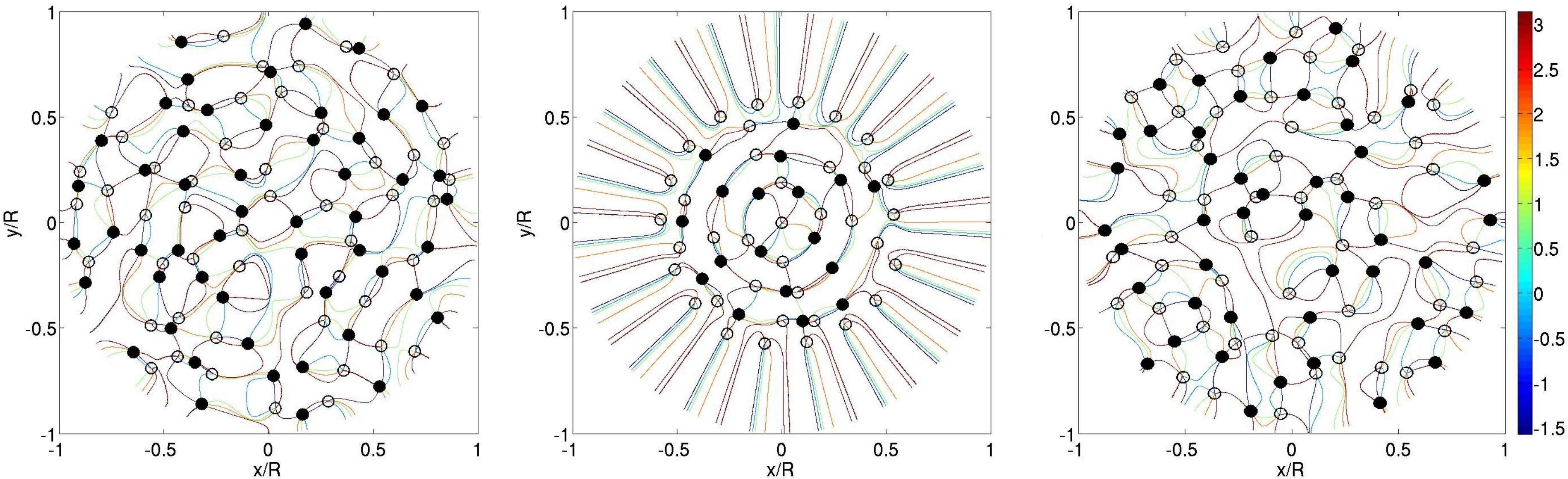
= superposition of the wave guide modes $\Psi = A_{mn}\, \Psi_{mn}$ with $\langle A_{mn} \rangle$ consistent with C_Ψ

Frequency away from cut-off

Cut-off frequency $\omega = \omega_{19,0}$

$\ell > 10^6 R$

$\ell = 10R$



Q=-2

=> $Q^2=4$

Q=19

=> $Q^2=361$

Q=5

=> $Q^2=25$

Conclusion

Fluctuation of charge in a wave guide:

- Sreening
- Cutt off frequencies => giant fluctuations of the topological charge
- Probe the mean free path even when it is much larger than the wave guide size

Other interest of phase:

You are welcome at the poster session to discuss about phase statistics and correlation and the possible applications to seismology.

Thank you for your attention


CENTRE NATIONAL
DE LA RECHERCHE
SCIENTIFIQUE
UNIVERSITE
JOSEPH FOURIER
SCIENCES TECHNOLOGIE MEDECINE
Congrès Général de la société française de Physiquue, Grenoble, 9-13 juillet 200