Anderson Localization
of
Light and Ultrasound

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laboratoire de physique et de modélisation des milieux condensés

dépasser les frontières

Université Joseph Fourier – Grenoble 1 / CNRS

Contents of the course

1. **Introduction to localized waves**
    - Historical perspective
    - Diffusion of waves and mean free path
    - Localization in a nutshell

2. **Theories of Localization**
    - Random Matrix theory
    - Ab-initio methods
    - Supersymmetric theory
    - Selfconsistent theory of localization

3. **Mesoscopic transport theory**
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Contents of the course

5. Enhanced backscattering as a precursor of strong localization

- Reciprocity principle
- Observation of enhanced backscattering of light and sound
- Return probability in infinite and open media

6 Speckles and correlations in wave transport

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- Short and long-range correlations

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– Historical perspective
– Recent experiments and link with localization

8. Observation of Anderson localization in high dimensions

- 3D light localization
- 2D transverse localization
- Quasi 1D localization of microwaves
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  - dynamics in transmission
  - transverse confinement
  - selfconsistent theory
  - speckle distribution
  - multifractal wave function

Contents of the course

5. **Anderson localization of noninteracting atoms in 3D**
    - Self-consistent Born approximation in speckle potential
    - Phase diagram from selfconsistent theory
    - Energy distribution of atoms

I. Introduction to Anderson Localization

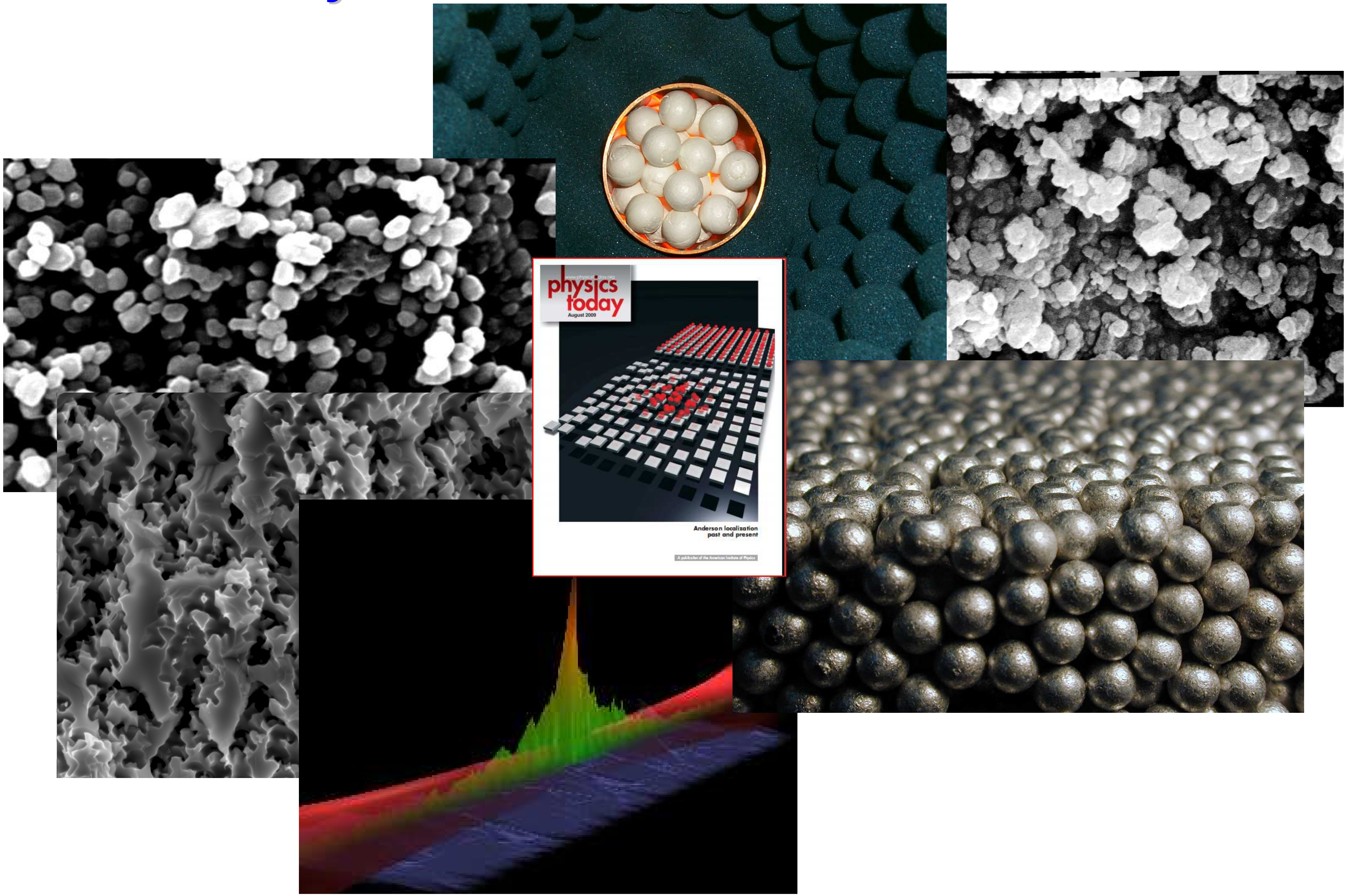
50 years of Anderson localization

Localization [..] very few believed it at the time, and even fewer saw its importance, among those who failed was certainly its author.
It has yet to receive adequate mathematical treatment,
and one has to resort to the indignity of numerical simulations to settle even the simplest questions about it.

P.W. Anderson, Nobel lecture, 1977

…..and now we have (numerical) experiments !

Physics Today, August 2009
50 years of Anderson Localization



Yes 

- Anderson's 1958 paper has « *often been quoted but hardly ever read* »
  and even less understood
  Several of his results have become mathematical theorems
- Anderson localization has become an « unrecognizable monster »
- Weak localization is (just) a precursor of strong localization
- P.W. Anderson created condensed matter physics

No 

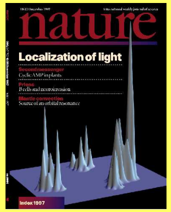
- Anderson's definition is zero diffusion constant
- Anderson shows how wave loops induce localization
- A local defect in a metal induces Anderson localization
- Anderson transition is discontinuous

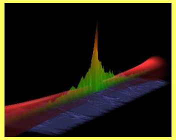
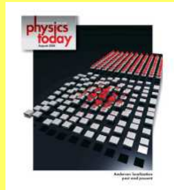
Maybe 

- 90 % of the publications is either wrong, trivial,
  already shown by Anderson, or not relevant
- Anderson localization would never have been found « accidentally »
  in a computer simulation
- After this lecture you will finally understand Anderson localization

1958	Anderson	« vanishing of diffusion »
1960	Mott/Ioffe-Regel	$\ell \leq \frac{\lambda}{2\pi}$
1965	Mott	Minimum conductivity Variable range hopping
1972	Thouless	Sensitivity to BC: g < 1 (Thouless criterion)
1973	Abou-Chacra/ Anderson/Thoules	Anderson model on the Caley Tree
1977	Anderson/Mott	Nobel Prize
1980	« gang of four »	Scaling theory of open media $\frac{\partial \log(g)}{\partial \log(L)}$
1980	Götze, Vollhardt, Wölfle	Self-consistent transport theory
1982	Halperin, Pruisken	Scaling theory of Quantum Hall effect
>1982	Sharvin, Lagendijk, Maret, Maynard,…	Weak localization Mesoscopic physics!

1983	**Fröhlich & Spencer**	**Mathematical proof for 3D Anderson model**
1984	**Anderson**	**25 years localization « *unrecognizable monster* »**
1986	**Anderson**	**« Theory of white paint »**
1986	**Kramer, Mackinnon, Economou, Soukoulis, Schreiber**	**Tight binding model and numerical Scaling**
1987	**Papanicolaou, Sheng**	**Prediction of Localization of Seismic Waves in layered Earth Crust**
1987	**Souillard**	**Localization of Gravitational Waves in Universe?**
1988	**John**	**Prediction of Localization of light in Photonic crystals**
1990	**Dorokhorov, Mello etal, Beenakker Altschuler**	**DMPK equation for wire interactions**

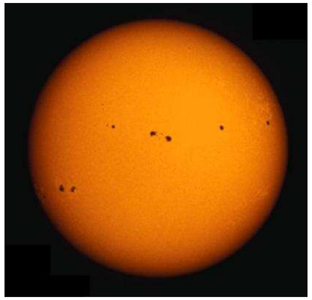
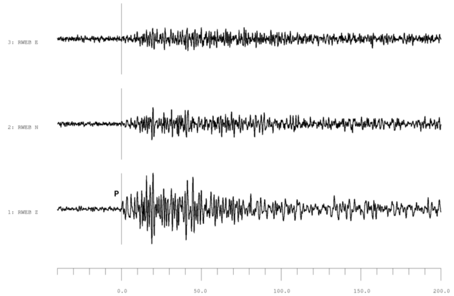
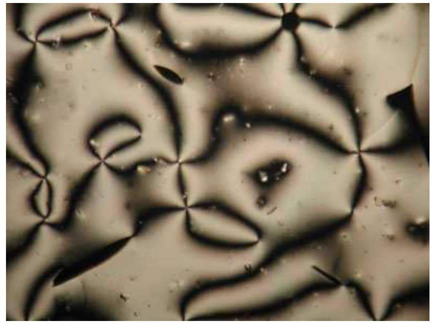
1992	Bell-labs Weaver Genack Exxon (Sheng etal)	2D localization microwaves 2D localization of ultrasound Q1D localization microwaves 2D localization of bending waves
1991	Lawandy	Observation of laser action in random media (threshold and line narrowing)
>1995	BEC community	Localization of light in BEC cold atomes gazes or the contrary?
1997	Wiersma, Lagendijk	3D localization of infrared light 
> 1998	Cao, Wiersma etal. Sebbah, …	Random lasering from (pre) localized states
2000	Genack Beenakker, … Van Tiggelen/ Lagendijk/Wiersma	Statistics in localized regime (exp) Idem (theo) D(r) in localized regime
2006	Maret	Anomalous dynamic transmission of light near mobility edge
2007	Fishman/Segev	Transverse Light Localization in 2D lattices

2008	Germinet, Klein etal (Cergy-Pontoise/ Irvine)	Proof of delocalized states near 2D Landau levels
2008	Palaiseau group Florence group	1D localization of noninteracting cold atoms 
2008	Lille group/ LKB	3D dynamical localization of cold atoms
2008	Winnipeg/Grenoble	3D localization of ultrasound 
2008	Cambridge UK IHP Paris	50 years of Anderson localization https://www.andersonlocalization.com 

2011	**Urbana Champaign**	**3D localization of fermions**
~~2012~~ 2011	**Palaiseau group**	**3D Localization of cold bosons**
2013		**4D kicked rotor with cold atoms**
2012		**Localization of light in BEC**
2013		**Localization of seismic waves in the Earth crust of La Réunion**
2014	**VIRGO**	**Non-observation of gravitational waves. They are localized!…..**

Localization of Waves
Review « Les Houches »,
http://lpm2c.grenoble.cnrs.fr/Themes/tiggelen/cv.html

Diffusion of Waves



Diffusion = random walk of waves

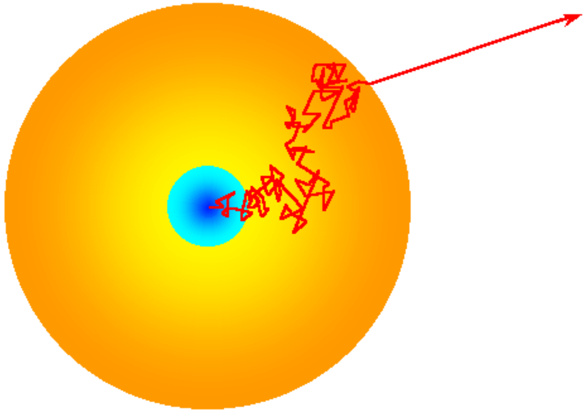
$$\partial_t \rho(\mathbf{r},t) - D\ \nabla^2 \rho(\mathbf{r},t) \pm \frac{\rho(\mathbf{r},t)}{\tau_{\text{abs,gain}}} = S\ \delta(t)\ \delta(\mathbf{r}-\mathbf{r}_S)$$

$$\langle \mathbf{r}^2(t) \rangle = \frac{\langle \rho(\mathbf{r},t)\mathbf{r}^2 \rangle}{\langle \rho(\mathbf{r},t) \rangle} = 6D\,t$$

$$D = {}^{1}\!/_{3}\ \mathrm{v}\ \ell^*$$

diffusion constant

Multiple scattering of Waves



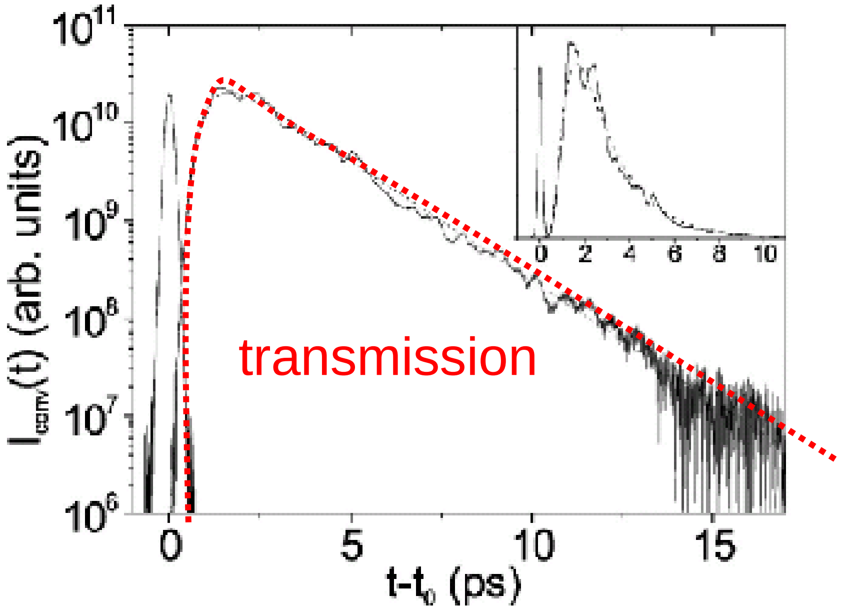
Mean free path $\boxed{\ell}$

« **particle language** »: **mean distance between successive scatterings**

« **Wave language** » $$\boxed{\langle \exp(i\phi) = 0 \rangle}$$

distance to randomise phase

Diffusion works even better than expected!



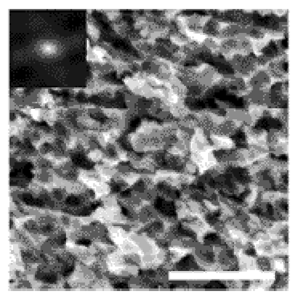
Diffusion equation

porous GaP

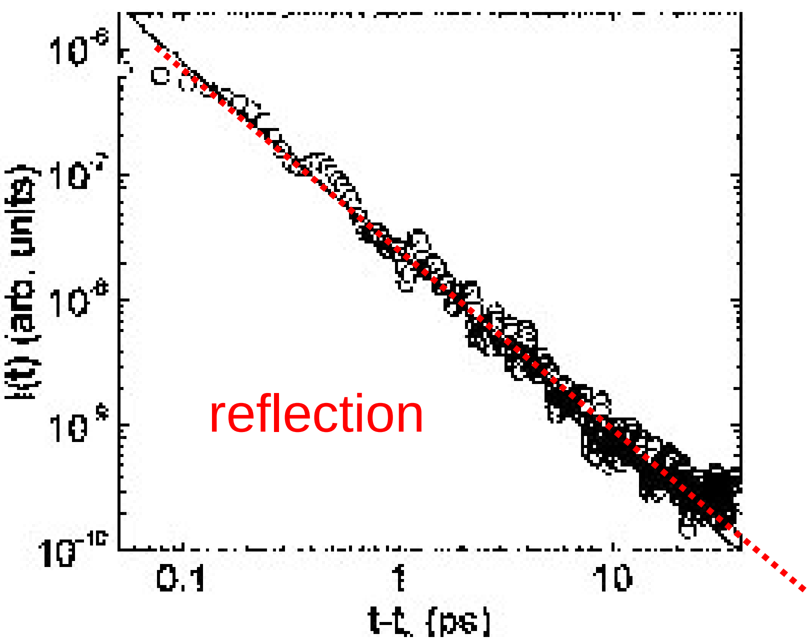
$$L = 20\ \mu m$$

$$\lambda = 739\ nm$$

Lagendijk etal, PRE 2003



1 µm



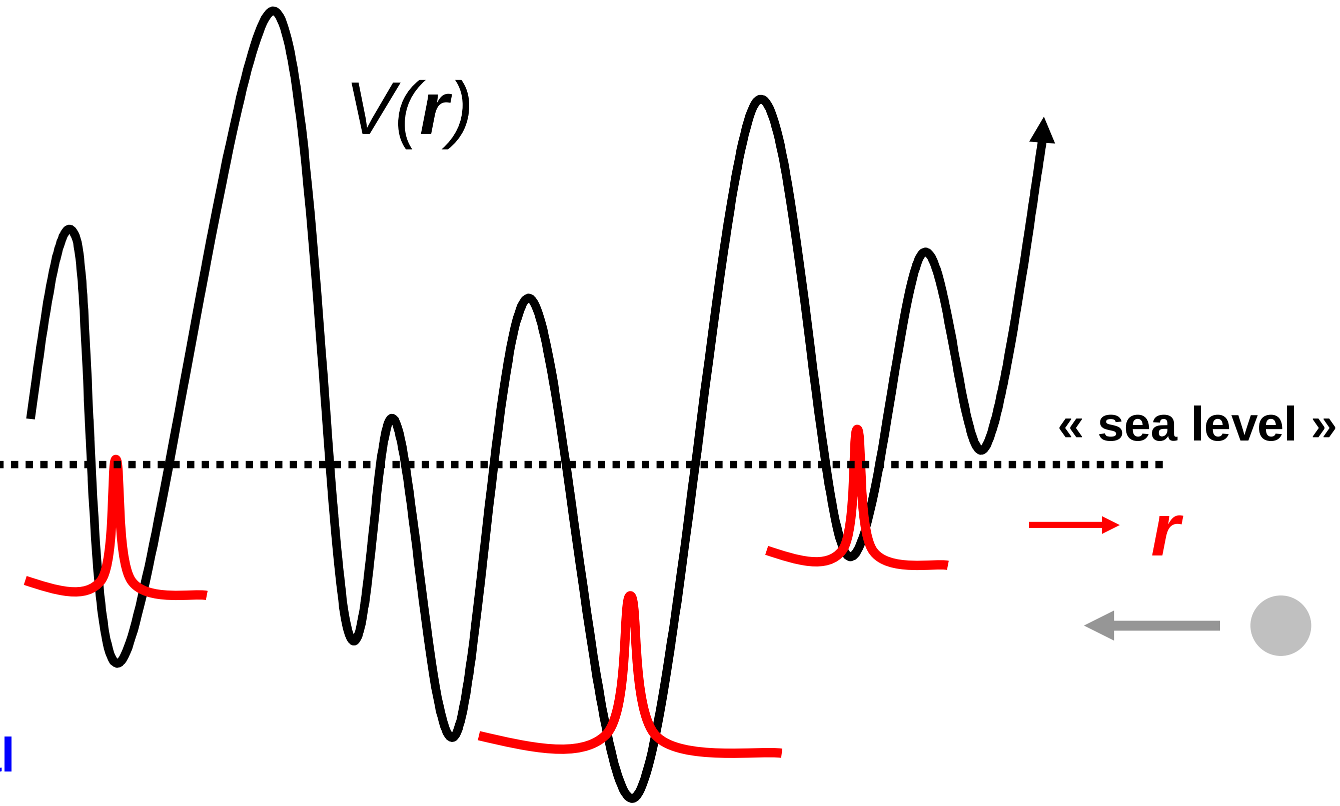
$1/t^{3/2}$

$$D = 23\ m^2 / s$$

$$\ell^* = 250\ nm \quad (k\ell^* = 2.1)$$

What is localization?

Dimension =1



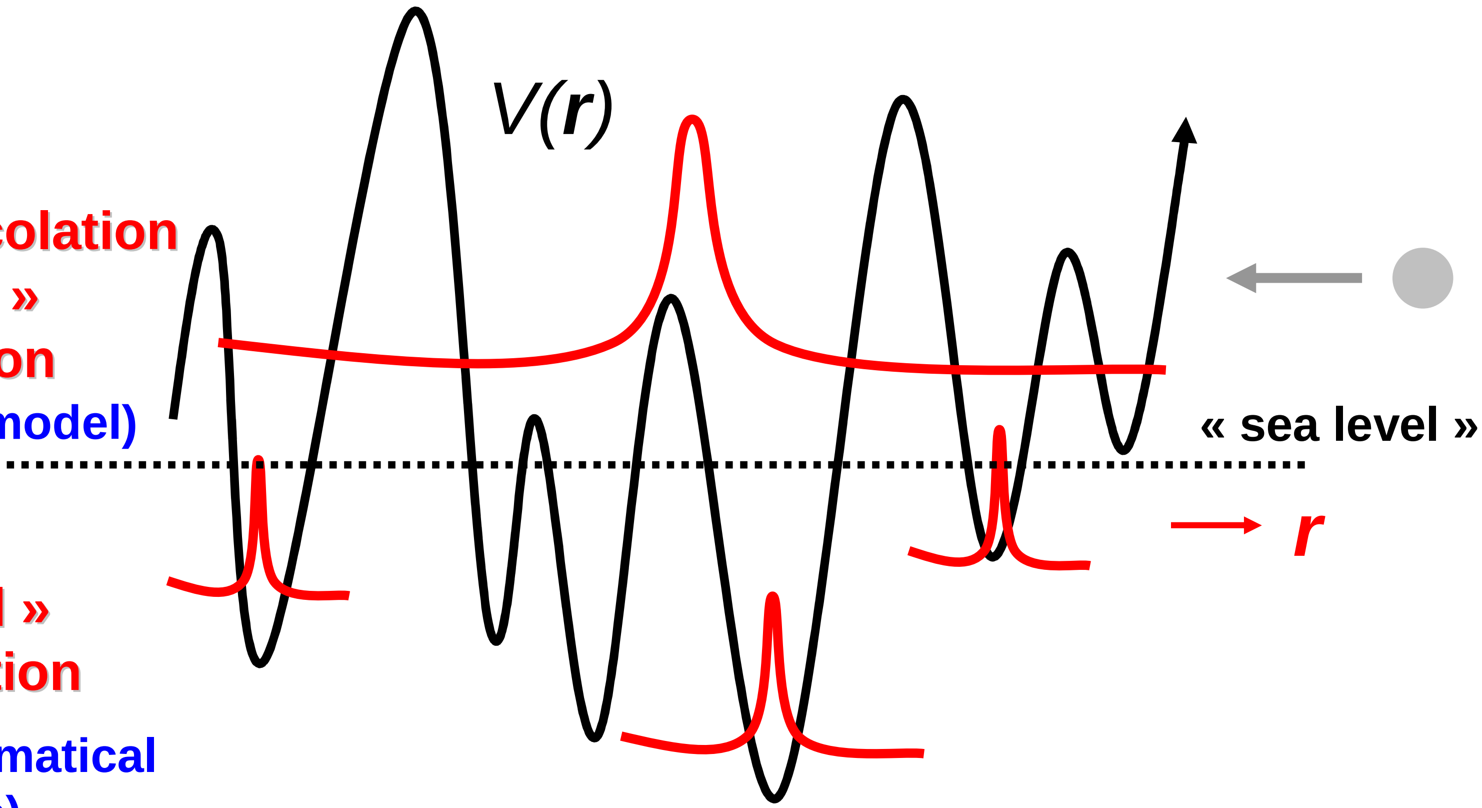
« Trivial » Localization

(most mathematical proofs)

Dimension =1

« Tunnel /percolation assisted » localization

(Anderson model)



« Trivial » Localization

(most mathematical proofs)

« genuine » Localization $E > V_{max}$

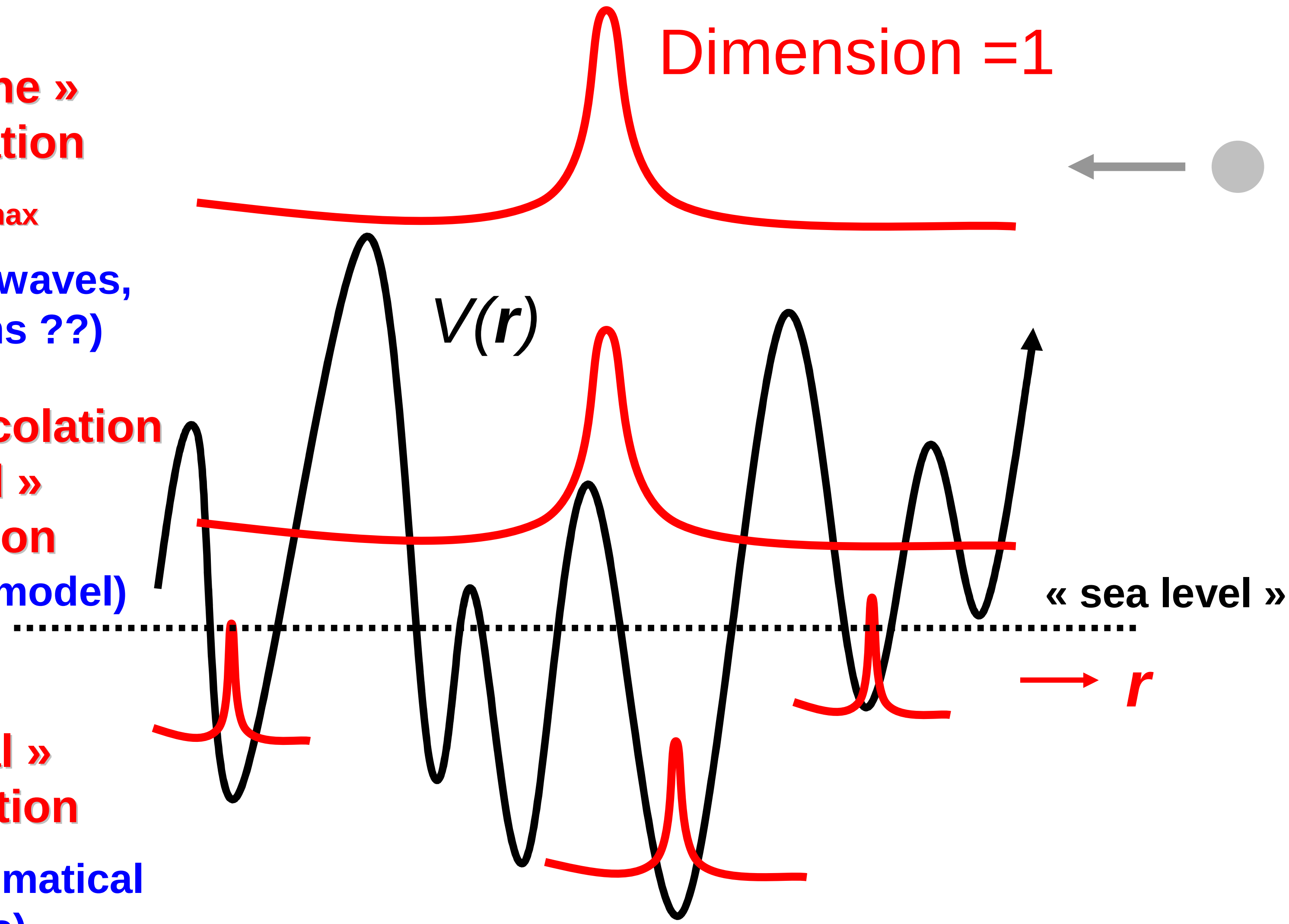
(Classical waves, cold atoms ??)

« Tunnel /percolation assisted » localization

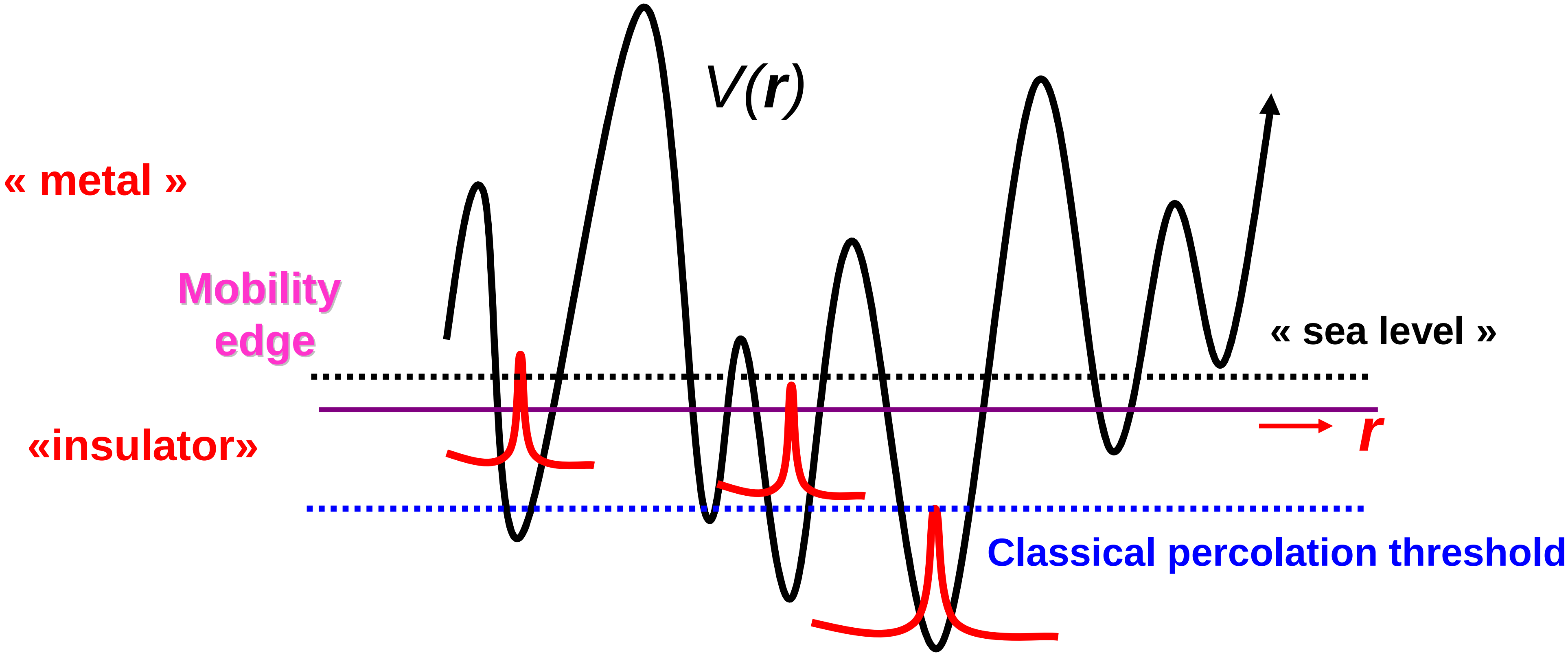
(Anderson model)

« Trivial » Localization

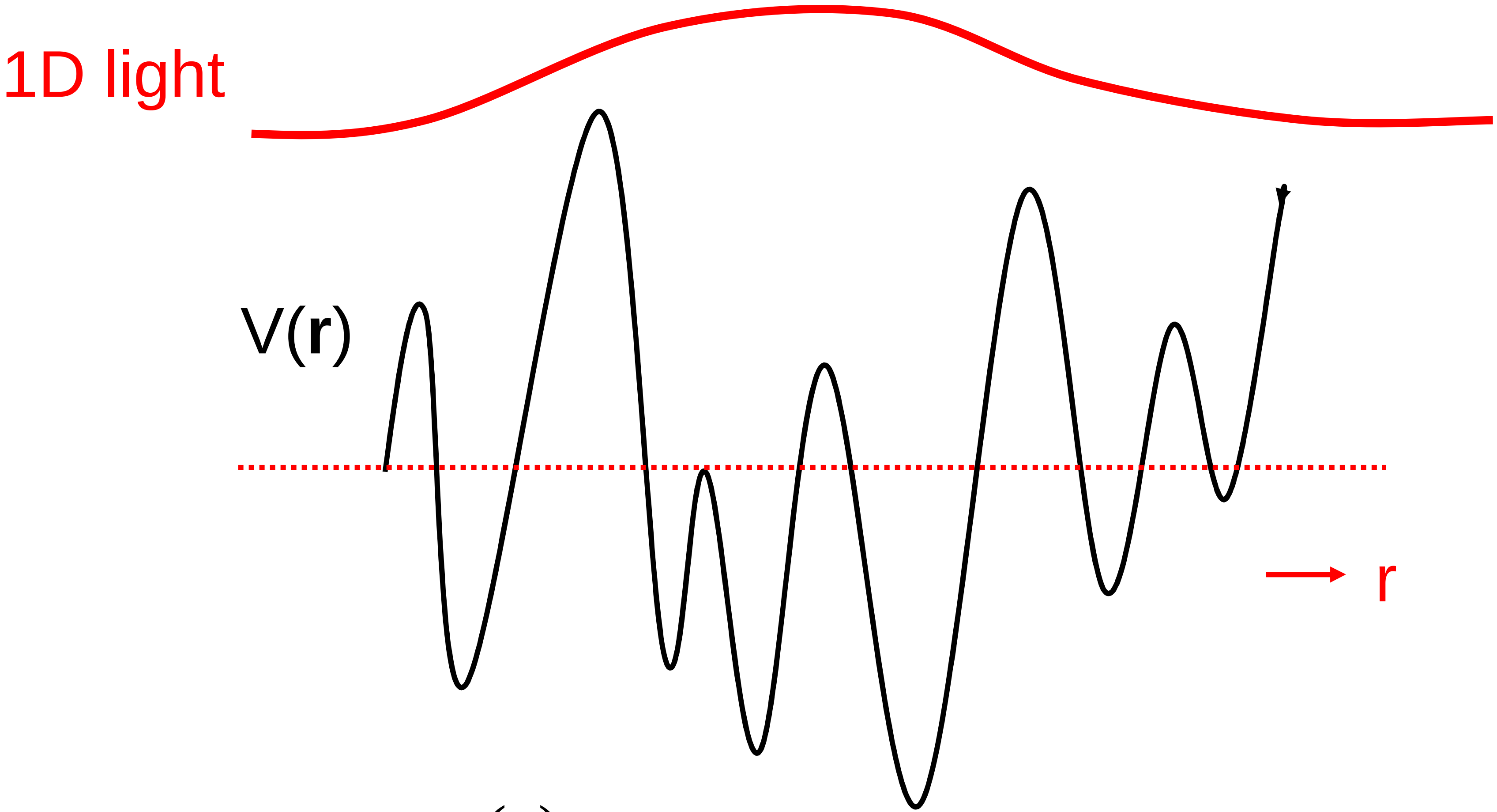
(most mathematical proofs)



Dimension = 3



V(**r**)
« metal »
Mobility edge
« sea level »
«insulator»
r
Classical percolation threshold



Classical waves

$$\frac{\varepsilon(\mathbf{r})}{c_0^2}\partial_t^2\psi(\mathbf{r},t)-\nabla^2\psi(\mathbf{r},t)=0$$

$$\psi=\psi(\mathbf{r})\exp(-i\omega t)\Rightarrow\begin{cases}V(\mathbf{r})=[1-\varepsilon(\mathbf{r})]\dfrac{\omega^2}{c_0^2}\\[2ex] E=\dfrac{\omega^2}{c_0^2}\end{cases}$$

$$E>V$$

Localization of classical waves is not trivial

« Unrecognizable monster ?»

Mott minimum conductivity

•Thouless criterion and scaling theory

•Quantum Hall effect

MIT and role of interactions

Tight binding model and loops

Chaos theory (DMPK equation)

Localization near band gaps

Full statistics of conductance and transmission

Random laser

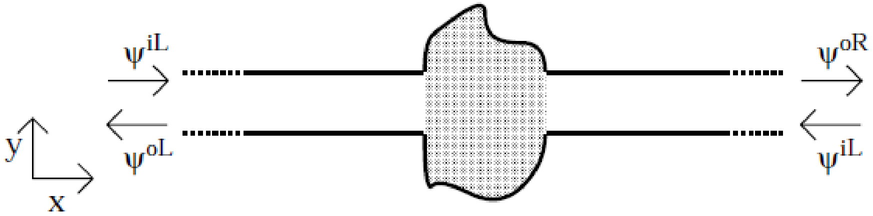
•Transverse localization

•Kicked rotor

II. Theory of Localization

IIa Random Matrix Theory

Elegant, and « nonperturbational » restricted to Q1D
'full counting statistics (Beenakker, RMP 1997)



Chaos theory of S-matrix

$$P(S) = \text{constant},$$

with respect to dS measure that respects symmetry

$$S = \begin{pmatrix} u & 0 \\ 0 & v' \end{pmatrix} \begin{pmatrix} \sqrt{1-T} & i\sqrt{T} \\ i\sqrt{T} & \sqrt{1-T} \end{pmatrix} \begin{pmatrix} u' & 0 \\ 0 & v \end{pmatrix}$$

$T_{nm}=T_n\delta_{nm}$

NxN diagonal transmission matrix

$$dS = du\, dv\, du'\, dv' \prod_{i<j} |T_i - T_j|^\beta \prod_{j=1}^{N} T_j^{-1+\beta/2} dT_j.$$

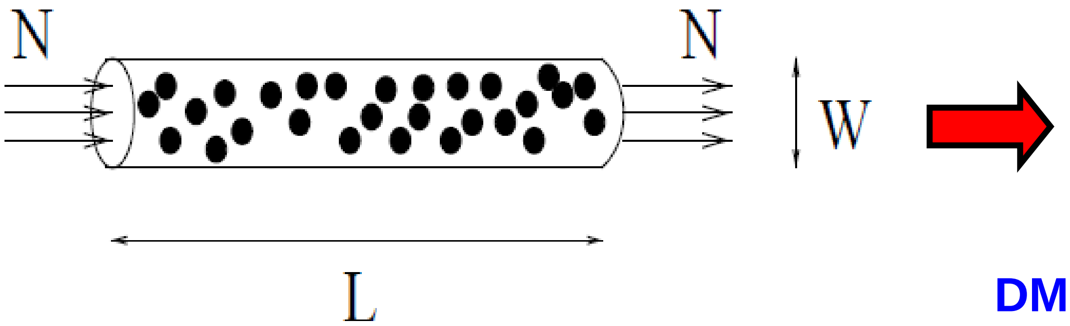
β=1,2,4

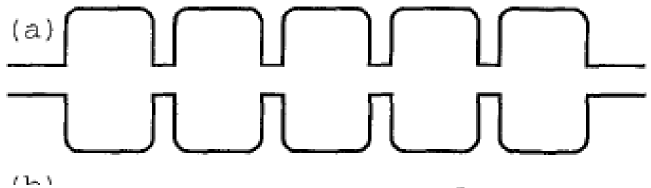
$$\langle G \rangle = \left\langle \frac{2e^2}{h} \sum_{n=1}^{N} T_n \right\rangle = \frac{2e^2}{h} \left[\frac{N}{2} + \frac{\beta - 2}{4\beta} + \mathcal{O}(N^{-1}) \right]$$

$$\text{Var } G/G_0 = \frac{1}{8}\beta^{-1},$$

Weak localization **UCF**

IIa Random Matrix Theory





DMPK Equation (Dorohhov 1982, Mello, Pereya, Kumar, 1988)

$$T_n = \frac{1}{1+\lambda_n}.$$

$$l\frac{\partial}{\partial L} P(\lambda_1, \lambda_2, \ldots, \lambda_N, L)$$

$$= \frac{2}{\beta N + 2 - \beta} \sum_{n=1}^{N} \frac{\partial}{\partial \lambda_n} \lambda_n (1+\lambda_n) J \frac{\partial}{\partial \lambda_n} \frac{P}{J}.$$

$$J(\{\lambda_n\}) = \prod_{i<j} |\lambda_i - \lambda_j|^{\beta}$$

$$\langle G \rangle = \left\langle \frac{2e^2}{h} \sum_{n=1}^{N} T_n \right\rangle = \frac{2e^2}{h} \frac{N\ell}{L}$$

$$\mathrm{Var}\ G/G_0 = \frac{2}{15}\beta^{-1}.$$

L < Nℓ: 'diffuse'

L > Nℓ: 'localized'

$$\langle g \rangle \approx \frac{16}{9} (2L/\pi\xi)^{-3/2} e^{-L/2\xi}.$$

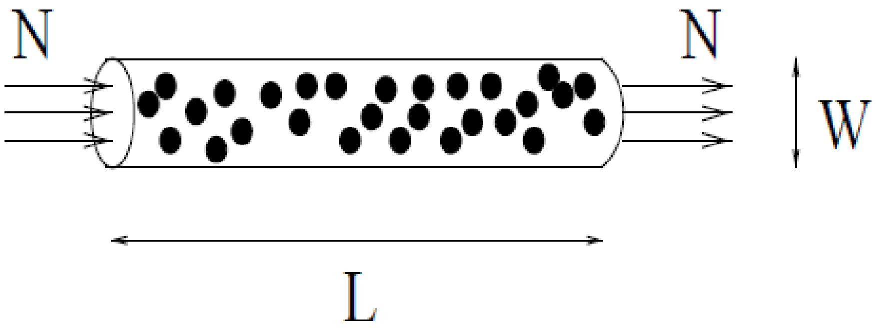
nonOhmic conductance

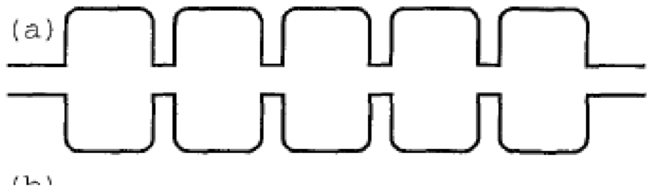
$$-\langle \ln(G/G_0) \rangle = \tfrac{1}{2}\ \mathrm{Var}\,[\ln(G/G_0)] = 2L/\gamma l$$

Lognormal distribution

$$\xi \approx \beta N l$$

IIa Random Matrix Theory





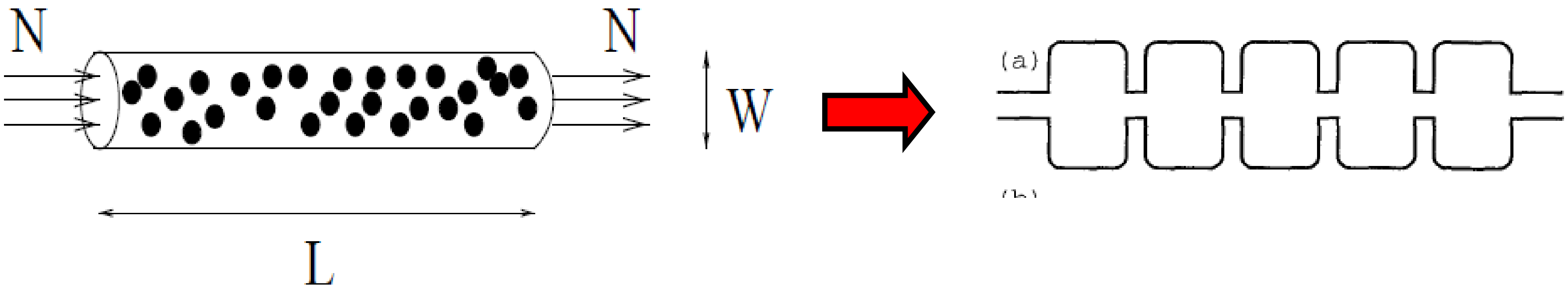
Eigenvalues of Transmission matrix T_{ab}

$$\rho(T) = \frac{1}{\pi\sqrt{T}\sqrt{1-T}},$$

Chaotic cavity

Bi-modal transmission !

IIa Random Matrix Theory



Eigenvalues of Transmission matrix T_{ab}

$$\rho(T) = \frac{1}{\pi\sqrt{T}\sqrt{1-T}} ,$$

Chaotic cavity

Bi-modal transmission !

Mello Kumar,1989

$$\rho(T) = \frac{\ell^*}{L}\frac{1}{2T\sqrt{1-T}} \qquad \text{for } 4e^{-2L/\ell^*} < T < 1 .$$

L < Nℓ: ‘diffuse’

Localized regime: necklace states exist with T=1 (Pendry, 1992)

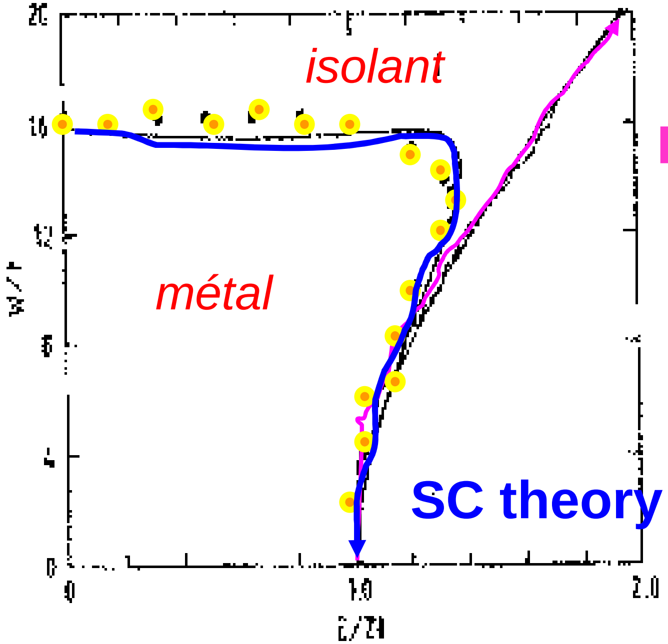
IIb. Ab initio (« exact solution of simplified model »)

Anderson Tight Binding Model (80-90), Random Dipoles 2000),
Large systems difficult (N < 5000)

$$H = \sum_{nn'} t \, |n\rangle\langle n'| \quad + \quad \sum_n V_n \, |n\rangle\langle n|$$

Random Disorder:

Disorder W



Energy E

Kroha, Wölfle, 1992

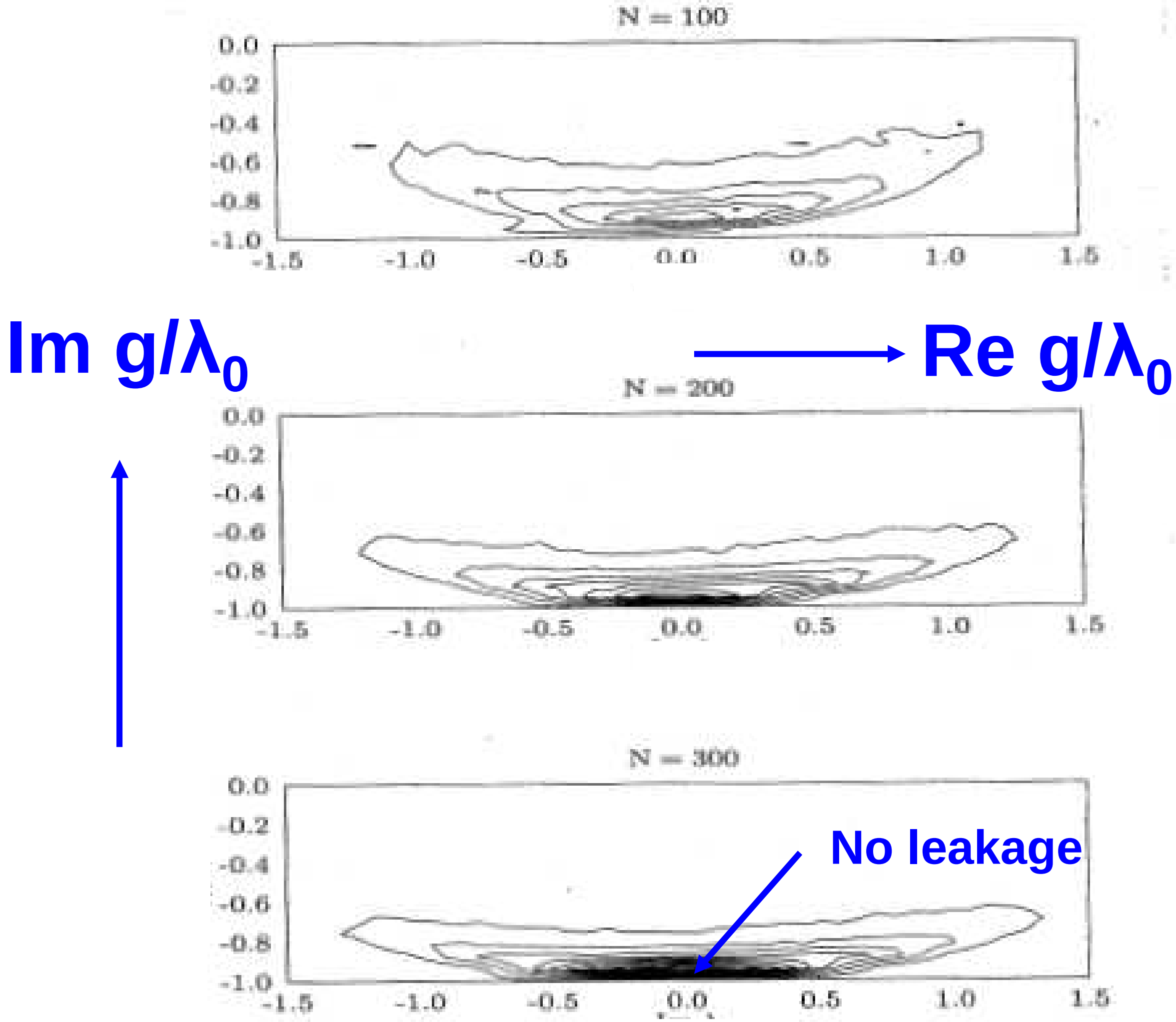
IIb. Ab initio (« exact solution of simplified model »)

$$\psi(\mathbf{r}_i) - \frac{4\pi\omega_0^3 / c_0^3}{\omega_0^2 - \omega^2 - i\gamma_0} \sum_{j=1}^{N} G(\mathbf{r}_i - \mathbf{r}_j)\,\psi(\mathbf{r}_j) = \psi_{in}(\mathbf{r}_i) := 0$$

$$G(r) = \frac{1}{4\pi r}\exp(ikr)$$

Eigenvalues of

$$\left[\left(\frac{\omega^2}{\omega_0^2} - 1\right) + i\lambda_0\right]\delta_{ij} + \underbrace{G(\omega r_{ij} / c_0)}_{g_{ij}}$$

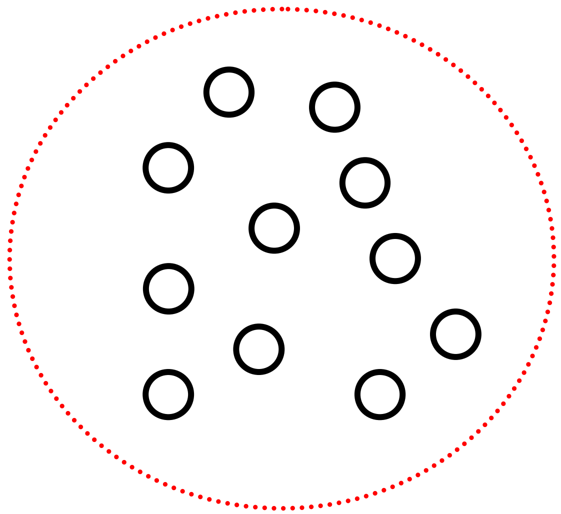


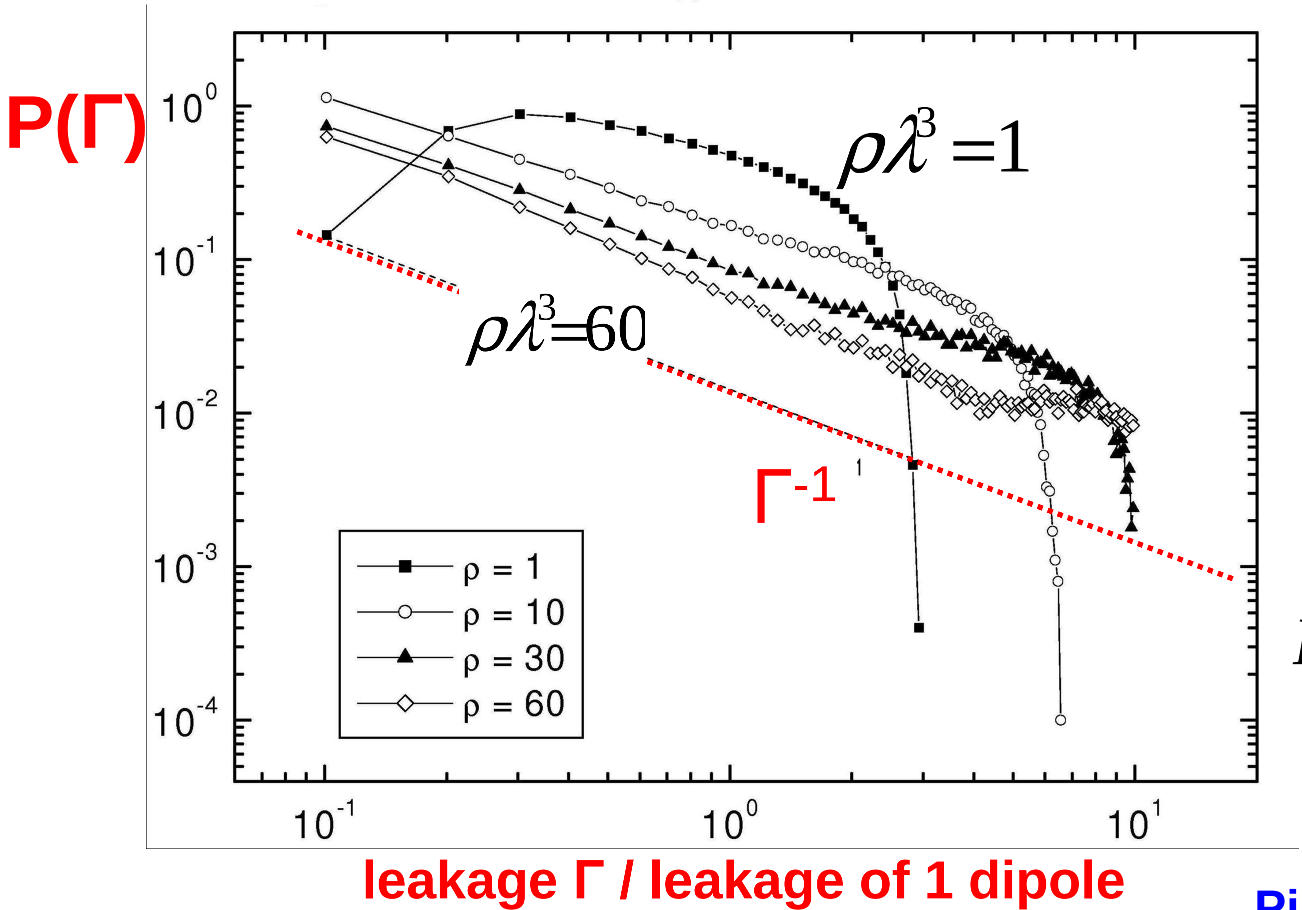
Rusek; Orlowski, 1997

IIb. Ab initio (« exact solution of simplified model »)

Random electric Dipoles (2000),

$$\psi(\mathbf{r}_i) = \psi_0(\mathbf{r}_i) + t \sum_{j \neq i}^{M} G(\mathbf{r}_{ij})\, \psi(\mathbf{r}_j) \,.$$





$$\Gamma(z) = \Gamma_0 |\psi(z)|^2$$
$$= \Gamma_0 \exp(-2z/\zeta)$$

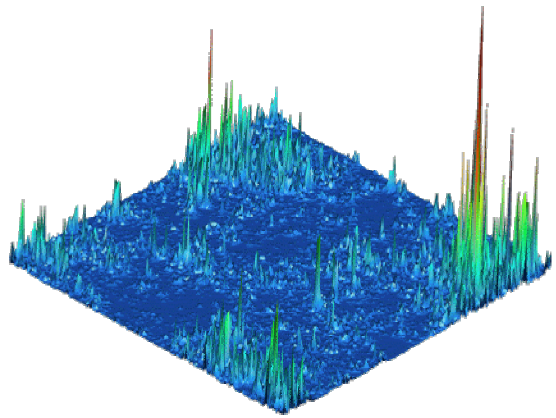
$$P(\Gamma) = \left|\frac{dn}{d\Gamma}\right| = \left|\frac{dn}{dz}\frac{dz}{d\Gamma}\right|$$
$$= \text{constant } \Gamma^{-1}$$

Pinheiro etal PRE 2004

IIc. Super symmetric field theory

- works for all dimensions
- nearly equivalent to DMPK in Q1D, also close to transition
- difficult to engineer with….

$$G^R_{E+\omega/2}(\mathbf{r_1},\mathbf{r_2})G^A_{E-\omega/2}(\mathbf{r_2},\mathbf{r_1})$$
$$= \int D\Phi\, D\Phi^\dagger S_1(\mathbf{r_1})S_1^*(\mathbf{r_2})S_2(\mathbf{r_2})S_2^*(\mathbf{r_1})$$
$$\times \exp\left\{i\int d\mathbf{r}\Phi^\dagger(\mathbf{r})[(E-\hat{H})\Lambda+\frac{\omega}{2}+i\eta]\Phi(\mathbf{r})\right\}$$



Inverse participation ratio

$$\langle P_q\rangle = L^d\langle|\psi(\mathbf{r})|^{2q}\rangle \sim L^{-\tau_q} \qquad \tau_q \equiv d(q-1)+\Delta_q,$$

Multifractality near mobility edge

$$\mathcal{P}(|\psi^2|) \sim \frac{1}{|\psi^2|}L^{-d+f(-\frac{\ln|\psi^2|}{\ln L})}$$

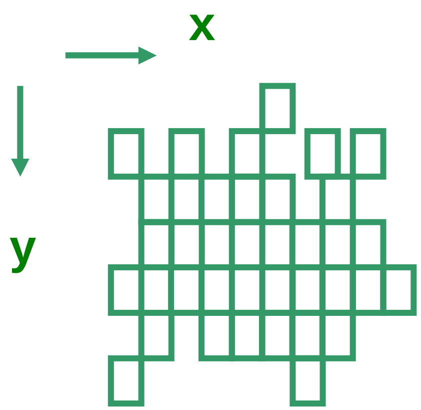
f(x): singularity spectrum

f(α) is the fractal dimension of the set of those points r where the eigenfunction intensity is $|\psi^2(r)| \sim L^{-\alpha}$.

Evers and Mirlin, RMP 2008

II.c Multifractality of wave function

Evers and Mirlin, Rev. Mod. Phys. (2008).



Box counting

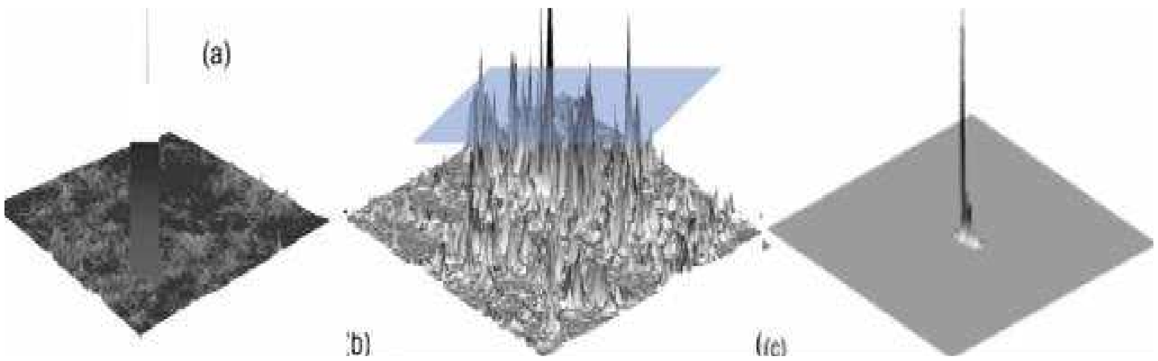
$$I_b = \frac{\int_{b^d} d^d\mathbf{r}\, I(\mathbf{r})}{\int_{L^d} d^d\mathbf{r}\, I(\mathbf{r})} \qquad \lambda << b << L$$

$$P_q = \sum_b (I_b)^q = \left(\frac{b}{L}\right)^{\tau_q}$$

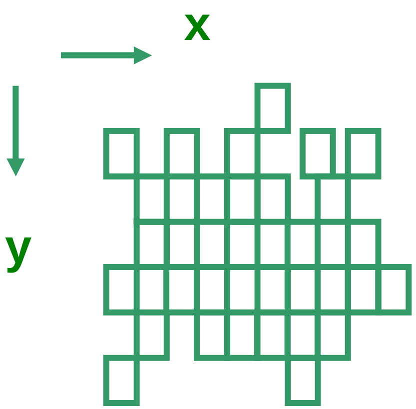
Generalized Inverse Participation Ratio

$$P(\log I_b) \propto \left(\frac{b}{L}\right)^{d - f\left(-\frac{\log I_b}{\log L/b}\right)}$$

Probability Distribution Function



Multifractality of wave function



Box counting

$$P(\log I_b) = \frac{1}{N_{L/b}} \left(\frac{b}{L}\right)^{d - f\left(-\frac{\log I_b}{\log L/b}\right)}$$

$$P_q = \sum_b (I_b)^q = \left(\frac{b}{L}\right)^{\tau_q}$$

$$\alpha \equiv \frac{\log I_b}{\log \lambda} \quad \lambda = \frac{b}{L} \downarrow 0$$

$$P_p = N\left\langle I_b^q \right\rangle = \lambda^{-d} \frac{1}{N_\lambda} \int d \log I_b \, I_b^q \, \lambda^{d - f\left(\log I_b / \log \lambda\right)}$$

$$= \frac{\log \lambda}{N_\lambda} \int d\alpha \lambda^{-f(\alpha)+q\alpha} \longrightarrow \begin{cases} \tau_q = \alpha^* q - f(\alpha^*) \\ q = f'(\alpha^*) \end{cases}$$

$$f(\alpha) = f(\alpha^*) + \tfrac{1}{2} f''(\alpha^*)(\alpha - \alpha^*)^2$$

Method of steepest descend

$$\tau_q = \underbrace{-d + dq}_{} + \Delta_q$$

Extended regime:

Anomalous gIPR

$$\Delta(q) = \gamma q(1-q) \Leftrightarrow f(\alpha) = d - \frac{1}{4\gamma}(\alpha - d - \gamma)^2$$

Lognormal PDF

Multifractality of wave function

Anomalous gIPR

Lognormal PDF

$$\Delta(q) = \gamma q(1-q) \Leftrightarrow f(\alpha) = d - \frac{1}{4\gamma}(\alpha - d - \gamma)^2$$

$$\tau_q =$$

$$d(q-1) + \quad \Delta_q^{(O)} = q(1-q)\epsilon + \frac{\zeta(3)}{4} q(q-1)(q^2-q+1)\epsilon^4$$

$$d(q-1) + \quad \Delta_q^{(U)} = q(1-q)(\epsilon/2)^{1/2} - \frac{3}{8} q^2 (q-1)^2 \zeta(3) \epsilon^2$$

(Wegner, 1987)

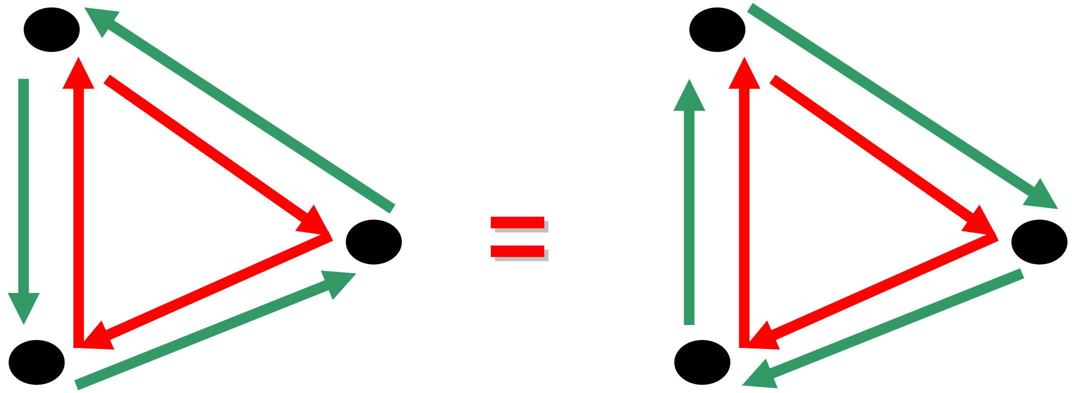
ε=d-2

IId. Selfconsistent theory of localization

- « Approximate solution of an exact model »
- Generalized diffusion equation, « simple »,
- Close to experiments, but mean field theory

(Vollhardt & Wölfle, 1980)

reciprocity



$$-\nabla \cdot D(\mathbf{r}) \nabla G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$$

$$\frac{1}{D(\mathbf{r})} = \frac{1}{D_B} + \frac{G(\mathbf{r},\mathbf{r})}{v_E \rho(\omega)\ell}$$

III. mesoscopic transport theory

III.a Dyson Green function

$$G(E,\mathbf{r},\mathbf{r}') = \left\langle \mathbf{r} \left| \frac{1}{E - \frac{p^2}{2m} - V(\mathbf{r}) + i0} \right| \mathbf{r}' \right\rangle \Rightarrow G(E,\mathbf{k},\mathbf{k}')$$

$$G(E,\mathbf{k},\mathbf{k}') = G(E,-\mathbf{k}',-\mathbf{k})$$ reciprocity

$$\langle G(E,\mathbf{k},\mathbf{k}') \rangle = G(E,\mathbf{k})\delta_{\mathbf{kk}'} := \frac{\delta_{\mathbf{kk}'}}{E - \langle V \rangle - \frac{\hbar^2 k^2}{2m} - \Sigma(E,k)}$$

Dyson Green function

$$\Sigma(E,k) = \Sigma(E) \Rightarrow G(E,k) = \frac{2m/\hbar^2}{\left[k(E) + \frac{i}{2\ell(E)}\right]^2 - k^2}$$

Self energy

Mean free path

k=1/2ℓ : strongly scattering

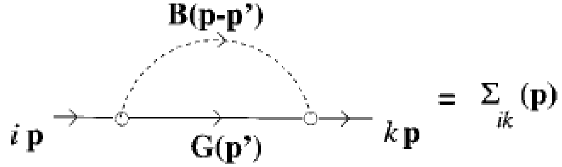
$$A(E,k) = -\frac{1}{\pi}\,\mathrm{Im}\,G(E,k)$$ Spectral function $$\int dE\;A(E,k) = 1$$

$$\rho(E) = \sum_{\mathbf{k}} A(E,k)$$ Average LDOS

Example 1: 3D white noise

$$\langle \delta V \rangle = 0 \;\; ; \langle \delta V(\mathbf{r}) \delta V(\mathbf{r}') \rangle = 4\pi U \; \delta(\mathbf{r}-\mathbf{r}') \;\; \Longrightarrow \;\; B(\mathbf{q}) = 4\pi U$$

$$G(p) = \frac{1}{E - p^2 - \Sigma} \;\; \Longrightarrow$$



$$\Sigma = 4\pi U \sum_{\mathbf{p}} \frac{1}{E - p^2 - \Sigma} = 4\pi U \sum_{\mathbf{p}} \left(\frac{1}{E - p^2 - \Sigma} + \frac{1}{p^2} \right) - 4\pi U \overbrace{\sum_{\mathbf{p}} \frac{1}{p^2}}^{K/4\pi}$$

$$= -iU\sqrt{E - \Sigma} - KU$$

$$\Rightarrow \Sigma(E) = \frac{1}{2}U^2 - KU - Ui\sqrt{E - (\tfrac{1}{4}U^2 - UK)}$$

$$\mathrm{Im}\,\Sigma = 0 \text{ for } E \le E_B = \tfrac{1}{4}U^2 - UK$$

$$E \ge E_B : G(E,p) = \frac{1}{E - E_B - \tfrac{1}{4}U^2 + iU\sqrt{E - E_B}} \equiv \frac{1}{(k + i/2\ell)^2 - p^2}$$

$$k = \sqrt{E - E_B}$$

$$\ell = \frac{1}{U}$$

$$k\ell \le 1 \Rightarrow E \le E_B + U^2 \qquad \ell = \left(\frac{\hbar^2}{2m}\right)^2 \frac{1}{U} \qquad E < E_B + \left(\frac{2m}{\hbar^2}\right)^3 U^2$$

Example 1: white noise

Spectral function

$$A(E,p) = \frac{1}{\pi} \frac{U\sqrt{E}}{\left(E - p^2 - \frac{1}{4}U^2\right)^2 + U^2 E}$$

Broad: $E^{-3/2}$

Energy distubution of cold atoms

$$N(E) = \sum_{\mathbf{p}} \Phi_\mu(p) A(E,p) \approx A(E,0)$$

$\mu \ll U^2$

Fraction of localized atoms

$$N_{loc} = \int_0^{U^2} dE\, A(E,0) = \frac{2}{\pi}\left(\arctan 2 - \frac{2}{5}\right) = 45\%$$

Example 2: discrete scatterers in d dimensions

$$\begin{cases} V(\mathbf{r}) = -\delta\varepsilon(\mathbf{r})\dfrac{\omega^2}{c_0^2} \\ E = \dfrac{\omega^2}{c_0^2} \end{cases}$$

$$V_{\mathbf{kk'}} = \int d\mathbf{r}\left(-\delta\varepsilon(\mathbf{r})\frac{\omega^2}{c_0^2}\right)\exp\left[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}\right] \approx -\delta\varepsilon\frac{\omega^2}{c_0^2}V_d$$

$$\Sigma_{\mathbf{kk'}}(\omega) = \left\langle\left(V_{\mathbf{kk'}} + \sum_{\mathbf{k''}} V_{\mathbf{kk''}}G(\omega,k'')V_{\mathbf{k''k'}} + \cdots\right)\exp\left[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}\right]\right\rangle_{\mathbf{r}}$$

$$= n\int d\mathbf{r}\, T_{\mathbf{kk'}}(\omega)\exp\left[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}\right] = nT_{\mathbf{kk}}\delta_{\mathbf{kk'}}$$

$$\frac{k}{\ell} = -\operatorname{Im}\Sigma_{\mathbf{k}}(\omega) = -n\operatorname{Im}T_{\mathbf{kk}}(\omega) = -n\sum_{\mathbf{k'}}\left|V_{\mathbf{kk'}}\right|^2\operatorname{Im}G(\omega,\mathbf{k}') + \cdots$$

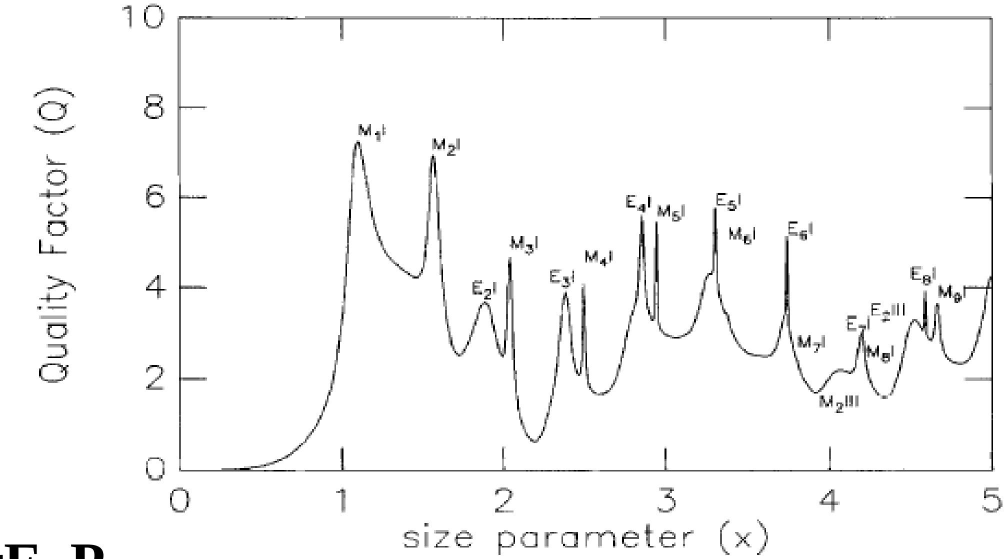
$$= n\pi\int \frac{d^d\mathbf{k}'}{(2\pi)^d}\left|V_{\mathbf{kk'}}\right|^2\delta\left(\frac{\omega^2}{c_0^2} - k'^2\right)$$

$$= nC_d(\delta\varepsilon)^2V_d^2\frac{\omega^4}{c_0^4}k^{d-2} \equiv kn\sigma$$

$$\Rightarrow \boxed{\sigma = C_d(\delta\varepsilon)^2V_d^2\frac{\omega^{d+1}}{c_0^{d+1}}}$$

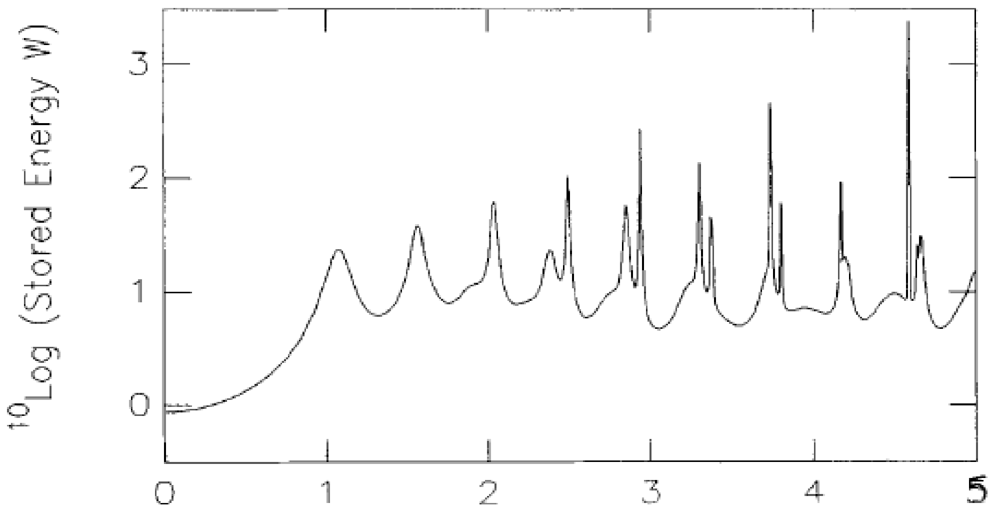
Mie sphere cross-section (3D)

$$Q = \frac{\sigma}{\pi a^2}$$



Elastic Cross-section

$$\frac{1}{\frac{4}{3}\pi a^3 E_{in}^2}\int_{\text{sphere}} d^3\mathbf{r}\mathbf{E}\cdot\mathbf{P}$$

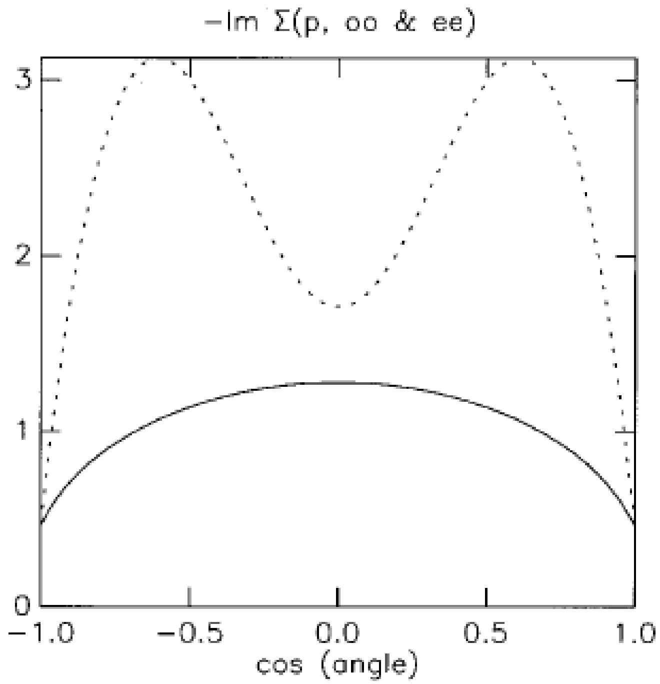


Stored energy in sphere

Example 3: nematic liquid cystal

$$B(\mathbf{f}) = \frac{(\omega/c_0)^4 \varepsilon_a^2 kT/K_1}{\mathbf{f}^2 + A(\mathbf{f}\cdot\mathbf{n})^2 + 1/\xi^2} \times \sum_{\alpha=1,2} (\mathbf{e}_\alpha \mathbf{n} + \mathbf{n}\mathbf{e}_\alpha)(\mathbf{e}_\alpha \mathbf{n} + \mathbf{n}\mathbf{e}_\alpha).$$

De Gennes, 1968



Mean free path depends on angle with optical axis and on polarization

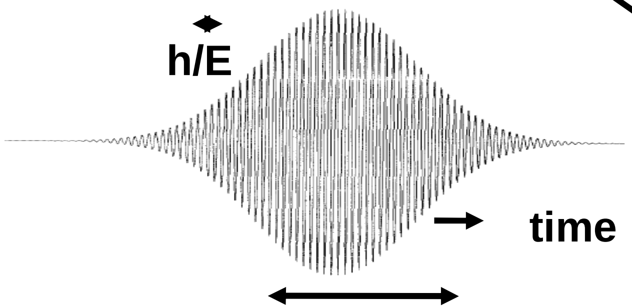
III.b Bethe-Salpeter equation

$$\left\langle G(E,\mathbf{r}_1,\mathbf{r}_1')G^*(E',\mathbf{r}_2,\mathbf{r}_2')\right\rangle \Rightarrow \left\langle G(E-\frac{\hbar\Omega}{2},\mathbf{k}-\frac{\mathbf{q}}{2},\mathbf{k}'-\frac{\mathbf{q}}{2})G^*(E+\frac{\hbar\Omega}{2},\mathbf{k}+\frac{\mathbf{q}'}{2},\mathbf{k}'+\frac{\mathbf{q}'}{2})\right\rangle$$

Two particle Green function

Momentum conservation

$$\Phi_{E\mathbf{kk}'}(\Omega,\mathbf{q})\delta_{\mathbf{qq}'}$$



$$\Phi_{E\mathbf{kk}'}(t,\mathbf{r})$$

Wigner function
(looks like phase space distribution)

$$\left|\Psi(\mathbf{r},t)^2\right| = \sum_{\mathbf{kk}'} \Phi_{\mathbf{kk}'}(\mathbf{r},t)\;\; S(\mathbf{k}')$$

$$\mathbf{J}(\mathbf{r},t) = \sum_{\mathbf{kk}'} \frac{\hbar\mathbf{k}}{m}\,\Phi_{\mathbf{kk}'}(\mathbf{r},t)\;\; S(\mathbf{k}')$$

Proba density

Proba current density

III.b Bethe-Salpeter equation

$\Phi_{E\mathbf{k}\mathbf{k}'}(t,\mathbf{r})$ obeyes a generalized transport equation (Bethe-Salpeter equation)

$$<GG^*>=<G><G^*>+<G><G^*>\otimes U\otimes<GG^*>$$

⇓

propagation

$$\left[-i\Omega+i\mathbf{k}\cdot\mathbf{q}+\Sigma(E+\tfrac{1}{2}\hbar\Omega,\mathbf{k}+\tfrac{1}{2}\mathbf{q})-\Sigma^*(E-\tfrac{1}{2}\hbar\Omega,\mathbf{k}-\tfrac{1}{2}\mathbf{q})\right]\Phi_{E\mathbf{k}\mathbf{k}'}(\Omega,\mathbf{q})$$
$$=A(E,k)\delta_{\mathbf{k}\mathbf{k}'}+A(E,k)\sum_{\mathbf{k}''}U_{E\mathbf{k}\mathbf{k}''}(\Omega,\mathbf{q})\Phi_{E\mathbf{k}''\mathbf{k}'}(\Omega,\mathbf{q})$$

Re: effective medium (mass renormalization)
Im: extinction

Source (initial condition)

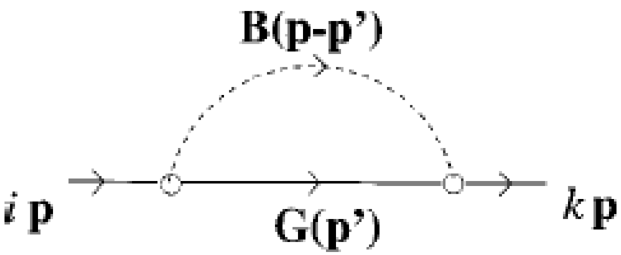
scattering

III.b Bethe-Salpeter equation

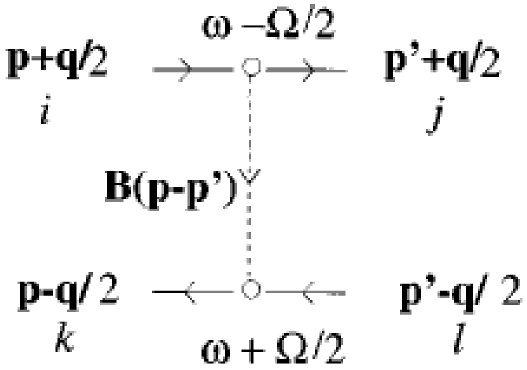
Born approximation: continuous media

$$\langle \delta\varepsilon_{ij}(\mathbf{r}) \rangle = 0 \qquad B_{ijkl}(\mathbf{q}) = \frac{\omega^4}{c_0^4} \int d\mathbf{r} \langle \delta\varepsilon_{ij}(\mathbf{r} - \tfrac{1}{2}\mathbf{x}) \delta\varepsilon_{kl}(\mathbf{r} + \tfrac{1}{2}\mathbf{x}) \rangle \exp(i\mathbf{q}\cdot\mathbf{x})$$

Structure factor



$$= \Sigma_{ik}(\mathbf{p}) \qquad \Sigma_{ik}(\omega,\mathbf{p}) = \int \frac{d^3p'}{(2\pi)^3} B_{ijkl}(\mathbf{p}-\mathbf{p}') G_{jl}(\omega,\mathbf{p}').$$



$$= U_{ijkl,\,\mathbf{p}\,\mathbf{p}'}(\mathbf{q},\Omega) = B_{ijkl}(\mathbf{p}-\mathbf{p}')$$

III.b Bethe-Salpeter equation

$$-i\Omega\sum_{\mathbf{k}} \Phi_{E\mathbf{k}}(\Omega,\mathbf{q}) + i\mathbf{q}\cdot\sum_{\mathbf{k}} \mathbf{k}\,\Phi_{E\mathbf{k}}(\Omega,\mathbf{q}) = S(E)$$

Continuity equation

provided

$$\Sigma(E+\tfrac{1}{2}\hbar\Omega,\mathbf{k}+\tfrac{1}{2}\mathbf{q}) - \Sigma^*(E-\tfrac{1}{2}\hbar\Omega,\mathbf{k}-\tfrac{1}{2}\mathbf{q}) = \sum_{\mathbf{k}''} U_{E\mathbf{k}\mathbf{k}''}(\Omega,\mathbf{q})$$

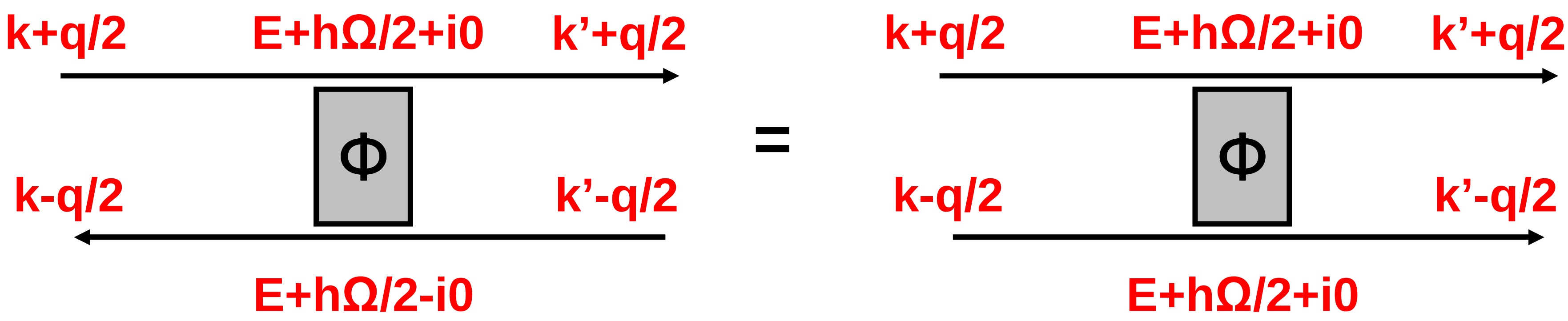
Ward identity

Probability is conserved

$$S(E) = \sum_{\mathbf{k}'} A(E,k')S(E,k')$$

Golden rule

Emitted power of source is determined by phase space distribution

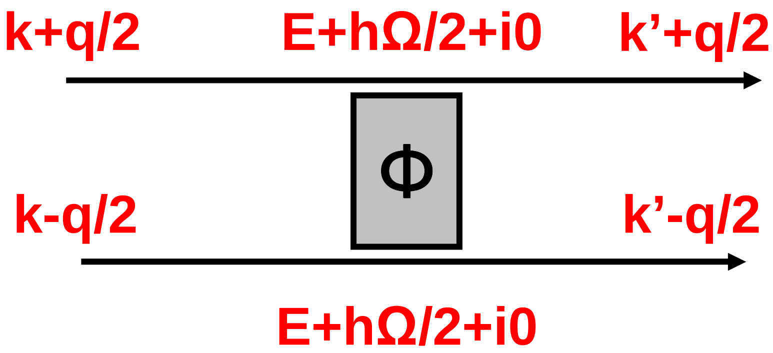


Feynman diagram
Real fields
bottom line « advanced » Green function
G(E-i0)

Scattering diagram
All retarded Green functions
G(E+i0)
Bottom line is complex conjugate

$$\langle \mathbf{k}|T(E-i0)|\mathbf{k}'\rangle = \langle \mathbf{k}'|T(E-i0)|\mathbf{k}\rangle^{*} = \langle \mathbf{k}'|T^{*}(E-i0)|\mathbf{k}\rangle^{*}$$
$$= \langle \mathbf{k}'|T(E+i0)|\mathbf{k}\rangle^{*}$$

Time reversal symmetry

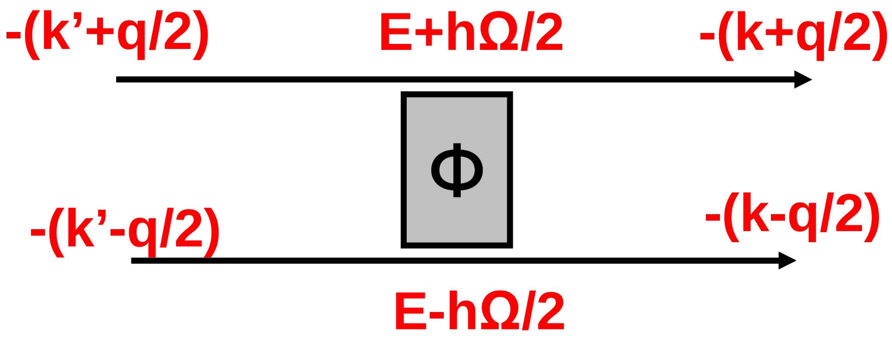


=

reciprocity

$$T_{\mathbf{kk'}}(E+i0)=T_{-\mathbf{k'}-\mathbf{k}}(E+i0)$$

1) =

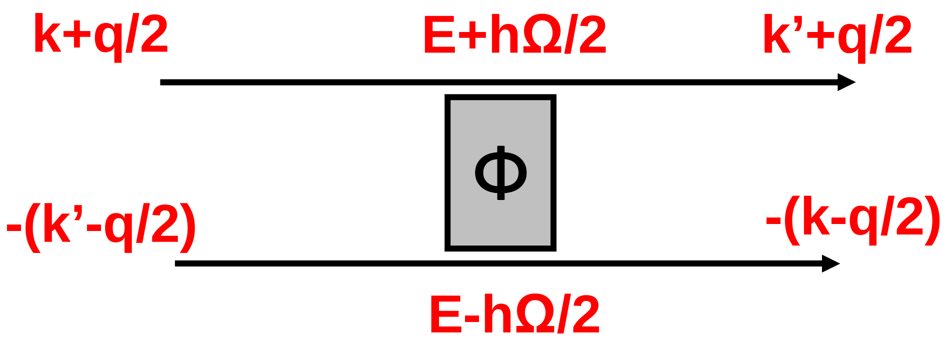


$$\Phi_{E\mathbf{kk'}}(\Omega,\mathbf{q})=\Phi_{E-\mathbf{k'}-\mathbf{k}}(\Omega,-\mathbf{q})$$

$$\Phi_{E\mathbf{k'k}}(\Omega,\mathbf{q})=$$

$$\Phi_{E\frac{\mathbf{k}-\mathbf{k'}+\mathbf{q}}{2}\frac{\mathbf{k'}-\mathbf{k}+\mathbf{q}}{2}}(\Omega,\mathbf{k}+\mathbf{k'})$$

2) =



III.c Diffusion approximation

$$\Omega \to 0, q \to 0$$

1th reciprocity

$$\Phi_{E\mathbf{kk}'}(\Omega,\mathbf{q}) = \frac{2\pi}{\rho(E)} \frac{\phi(E,\mathbf{k},\mathbf{q})\phi(E,\mathbf{k}',\mathbf{q})}{-i\Omega + D(E)q^2}$$

Proba of quantum diffusion

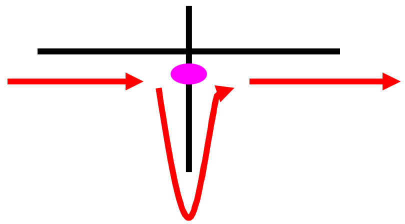
DOS normalization

$$\phi(E,\mathbf{k},\mathbf{q}) = A(E,k) - i\mathbf{k}\cdot\mathbf{q}\,F(E,k)$$

Equipartition in phase space

Small non-equilibrium current

$$\int d\mathbf{r} \sum_{\mathbf{k}} \Phi_{E\mathbf{kk}'}(t,\mathbf{r}) = \int \frac{d\Omega}{2\pi} \exp(-i\Omega t)\,\Phi_{E\mathbf{kk}'}(\Omega,\mathbf{q}=0)$$



$$= \frac{2\pi}{\rho(E)} \int \frac{d\Omega}{2\pi} \exp[-i\Omega t] \frac{\sum_{\mathbf{k}} A(E,k)A(E,k')}{-i\Omega} = 2\pi\, A(E,k') \quad (t>0)$$

$$\int \frac{dE}{2\pi}(..) = 1$$

Source weigted by spectral function

$$\Phi_{E\mathbf{k}\mathbf{k}'}(\Omega,\mathbf{q}) = \frac{2\pi}{\rho(E)} \frac{\phi(E,\mathbf{k},\mathbf{q})\phi(E,\mathbf{k}',\mathbf{q})}{-i\Omega + D(E)q^2}$$

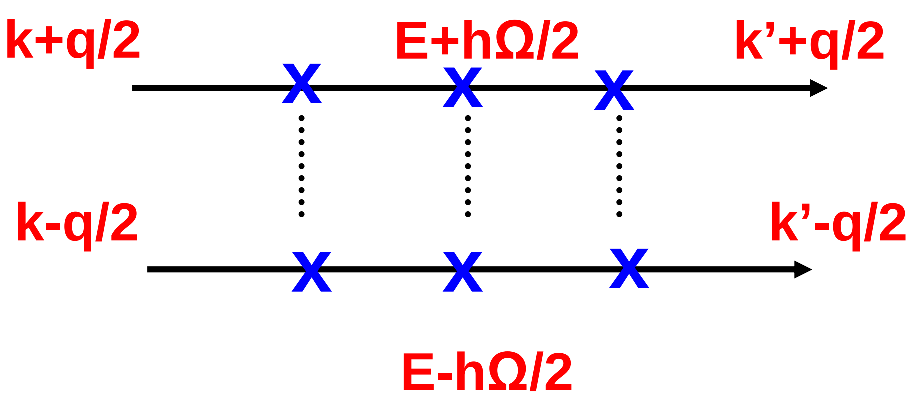
$$\phi(E,\mathbf{k},\mathbf{q}) = A(E,k) - i\mathbf{k}\cdot\mathbf{q}\,F(E,k)$$

$$D(E) = \frac{\hbar}{3m}\frac{1}{\rho(E)}\sum_{\mathbf{k}} k^2 F(E,k)$$

Kubo Greenwood formula

$$F(E,k) = \frac{2}{1-\langle\cos\vartheta\rangle}\mathrm{Im}^2\, G(E,p) + \cdots$$

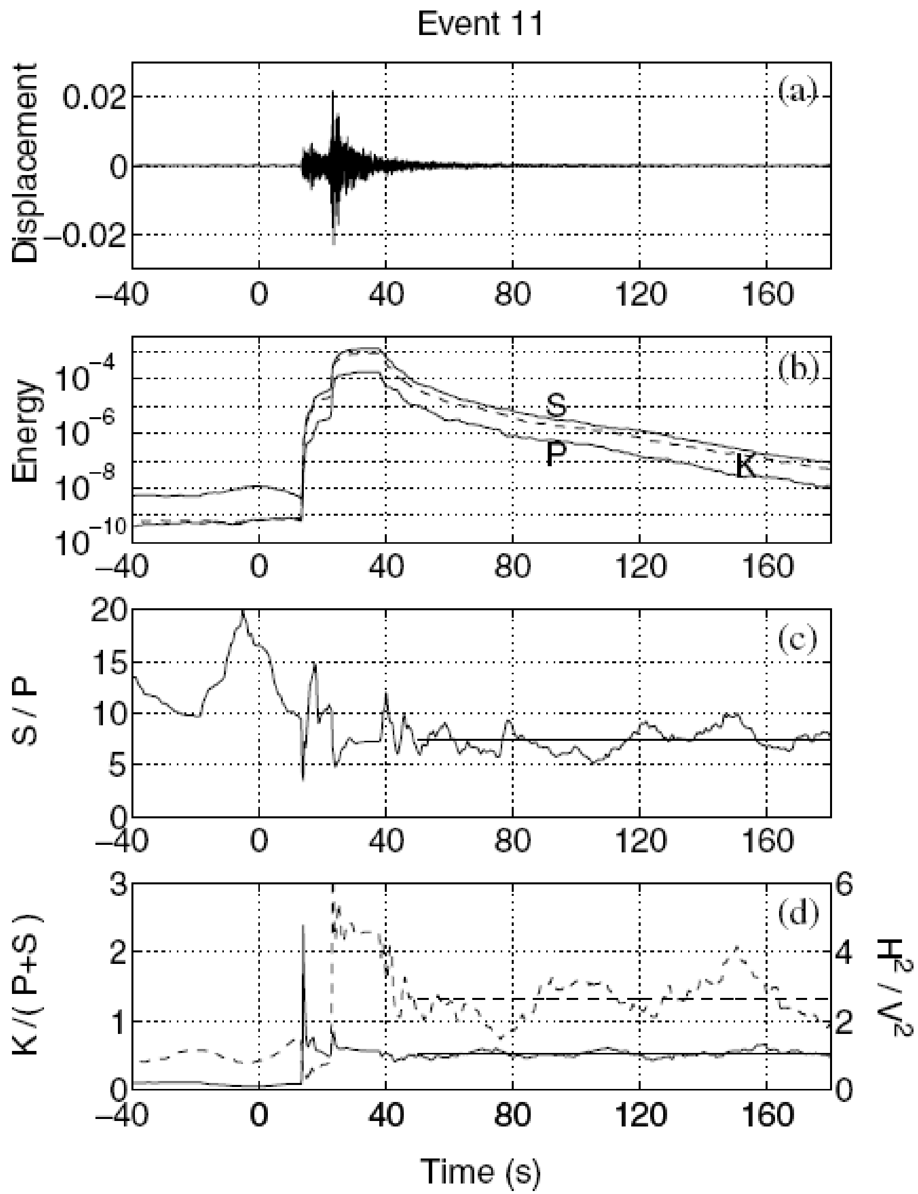
Boltzmann approximation



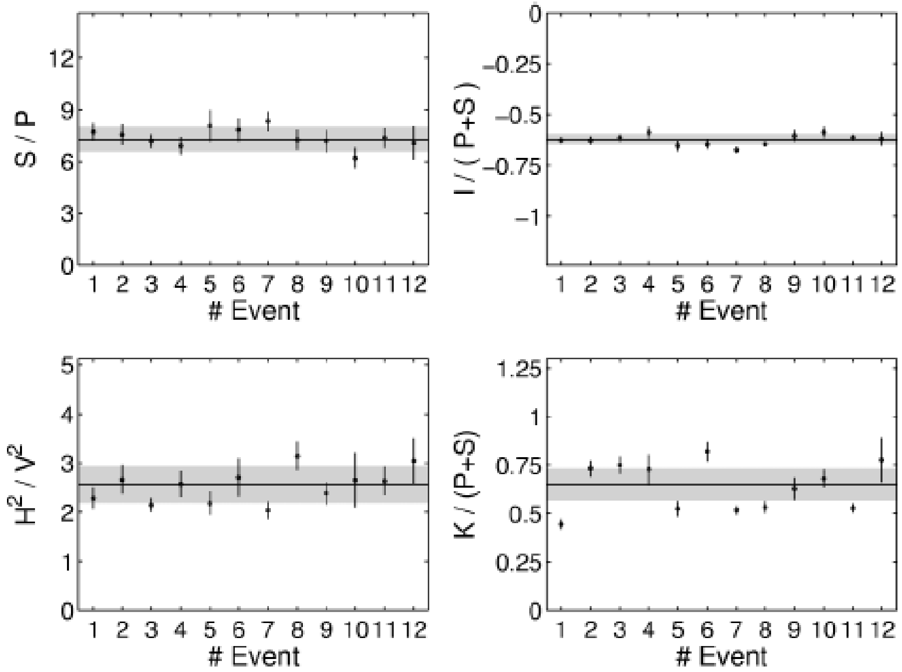
$$D(E) = \frac{1}{3}\mathrm{v}\frac{\ell}{1-\langle\cos\theta\rangle} = \frac{\hbar}{3m}\langle k(E)\rangle \ell^*(E)$$

Mahan, many particle physics

Equipartition of diffuse seismic waves in phase space



$$\rho_{P,S}(\omega) = \frac{g_{P,S}\,\omega^2}{2\pi^2 c_{P,S}^3}$$



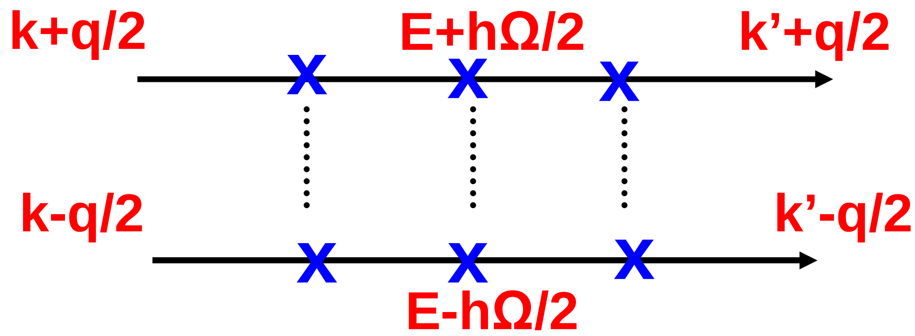
Hennino etal 2001

III.d Inclusion of interference

What about 2nd reciprocity? $\Phi_{E\mathbf{k}'\mathbf{k}}(\Omega,\mathbf{q}) = \Phi_{E\frac{\mathbf{k}-\mathbf{k}'+\mathbf{q}}{2}\frac{\mathbf{k}'-\mathbf{k}+\mathbf{q}}{2}}(\Omega,\mathbf{k}+\mathbf{k}')$

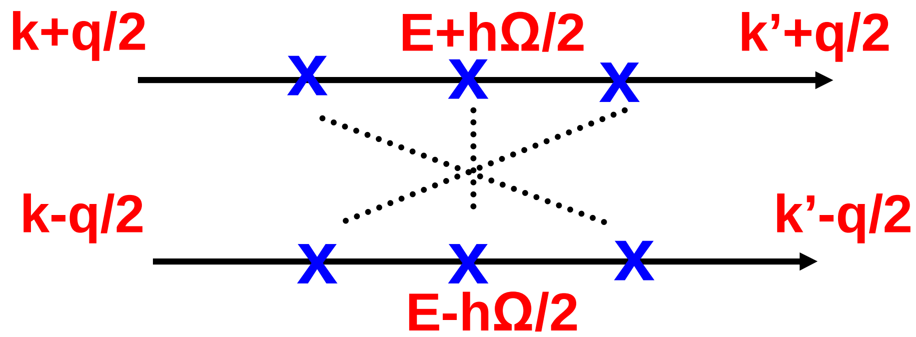
$$\Phi_{E\mathbf{k}\mathbf{k}'}(\Omega,\mathbf{q}) =$$

« ladder »



$$\frac{1}{-i\Omega + D(E)\mathbf{q}^2}$$

+



« most-crossed »

$$\frac{1}{-i\Omega + D(E)(\mathbf{k}+\mathbf{k}')^2}$$

Fully connected diagram: part of $U_{E\mathbf{k}\mathbf{k}''}(\Omega,\mathbf{q})$

III.d Inclusion of interference

Self-consistent theory of localization

$$\frac{1}{D(\omega)} = \frac{1}{D_B} + \frac{C_d}{\mathrm{v}\rho(\omega)\ell} \sum_{\mathbf{q}} \frac{1}{D(\omega)q^2}$$

DOS

return quantum probability
G(r,r)

$$E \leftrightarrow \frac{1}{2}\frac{(\hbar\omega)^2}{mc_0^2}$$

Vollhardt and Wölfle, 1980

$$D = \frac{1}{d}\mathrm{v}\ell^*$$

$$\ell^* \neq \ell$$

3D unbounded medium

$$G(\mathbf{r},\mathbf{r}') = G(\mathbf{r}-\mathbf{r}'); \qquad G(\mathbf{r},\ \mathbf{r}) = G(0) = \int_{q<1/\ell} d^3\mathbf{q} \quad \frac{1}{Dq^2}$$

$$\Rightarrow \boxed{D = D_B\left(1 - \frac{C_3}{\mathrm{v}\rho(\omega)\ell^2}\right)}$$

OK Mott:

$$\rho(\omega) \propto \frac{k^2}{\mathrm{v_F}} \qquad \rho(\omega) \approx \frac{k^2}{\mathrm{v_E}}$$

Electrons $E := \hbar\omega$ **light**

kℓ=1

→Localization of electrons at low energies

→Localization of light at frequencies with low DOS (near band gaps)

The erroneous (?) Mott argument for metal-insulator transition (1960)

$$\sigma(E) = e^2 \rho(E) D(E) = \frac{2\pi\gamma_d}{d} \frac{e^2}{h} k_F^{d-1} \ell^*$$

DOS per unit volume $= \gamma_d \dfrac{k_F^{d-1}}{\hbar \mathrm{v_F}}$

Diffusion constant $= \dfrac{1}{d} \mathrm{v}_F \ell^*$

$$\text{if } k\ell^* > 1: \sigma > e^2/h \;\; k_F^{d-2}$$

metal

$$\text{if } k\ell^* < 1: \sigma = 0$$

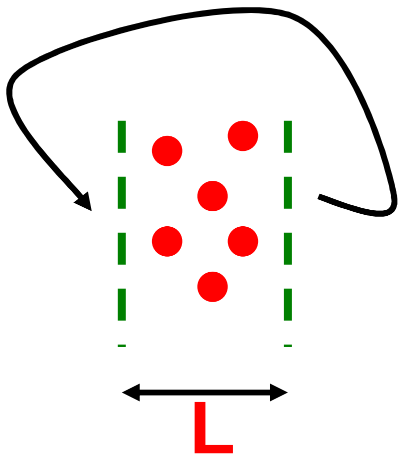
insulator

Quantum phase transition is discontinuous???

2D minimum conductivity ? $\sigma = \text{constant}\; e^2/h$

Slab with periodic BC's

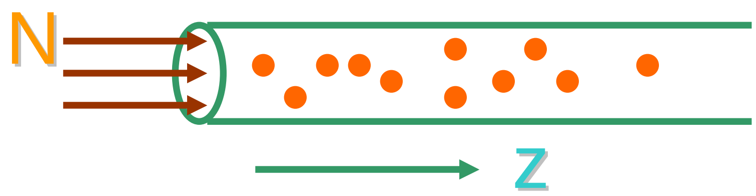
$$D(\mathbf{r}) = D \Rightarrow \quad G(\mathbf{r},\mathbf{r}) = \int_{q<1/\ell} d^2\mathbf{q} \; \frac{1}{D} \sum_{n \neq 0} \; \frac{1}{n^2\pi^2 / L^2 + q^2}$$



$$\Rightarrow \quad D(L) \approx D_B \left(1 - \frac{1}{(k\ell)^2} + \frac{\ell}{L} \right)$$

OK scaling theory

Quasi-1D wave guide



$$G(\mathbf{r},\mathbf{r}) = \frac{1}{A}G(z,z) \Rightarrow \frac{1}{D(z)} = \frac{1}{D_B} + \frac{2}{\xi}G(z,z) \qquad \xi = 2Ac_0\rho(\omega)\ell = 2N\ell$$

$$d\tau \equiv \frac{dz}{D(z)} \Rightarrow -\partial_\tau^2 G(\tau,\tau') = \delta(\tau-\tau') \Rightarrow G(\tau,\tau) = \quad \tau$$

$$\Rightarrow \frac{d\tau}{dz} := \frac{1}{D} = \frac{1}{D_B} + \frac{2}{\xi}\tau \Rightarrow \boxed{D(z) = D_B \exp\left(-\frac{2z}{\xi}\right)}$$

SCT

$$\frac{1}{T} = \mathrm{v}_E \int_0^L \mathrm{d}\tau = \mathrm{v}_E \int_0^L dz \frac{1}{D(z)} \propto \exp(L/\xi)$$

Difference?

DMPK:

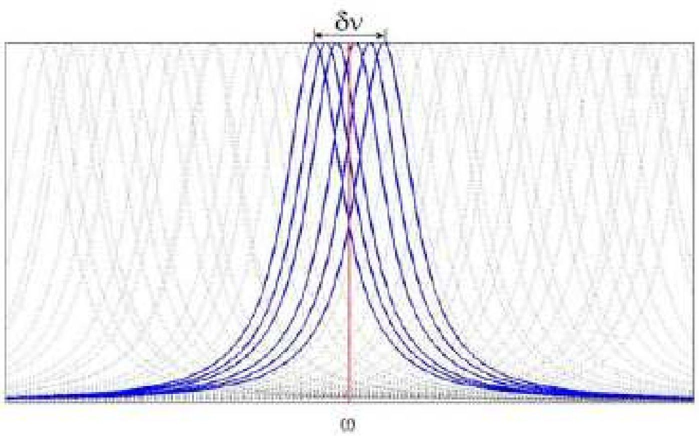
$$T \propto L^{-3/2} \exp(-L/2\xi)$$

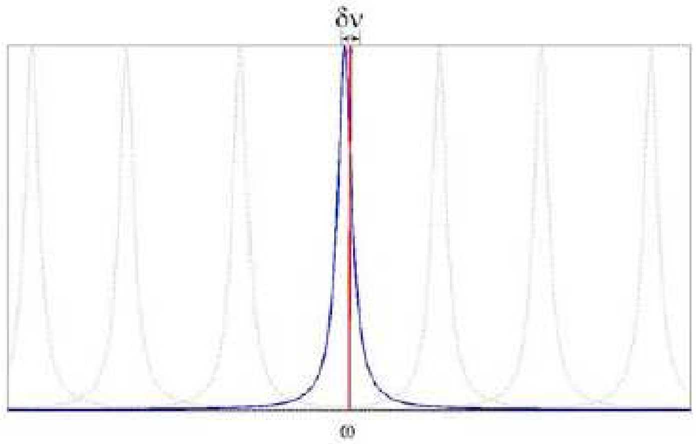
Quasi-1D wave guide

Level spacing $\delta\omega = \frac{D}{L^2} = \frac{c\ell}{L^2}$ $\Delta\omega = \frac{1}{N}\frac{c}{L}$ **broadening**

$$\frac{\Delta\omega}{\delta\omega} = \frac{L}{N\ell} = \frac{L}{\xi}$$

>1 **<1**



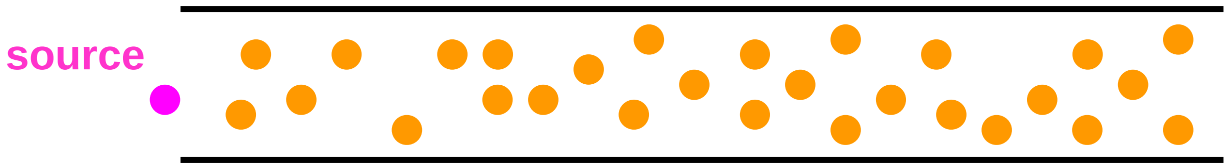


Continuous spectrum
Transport theory

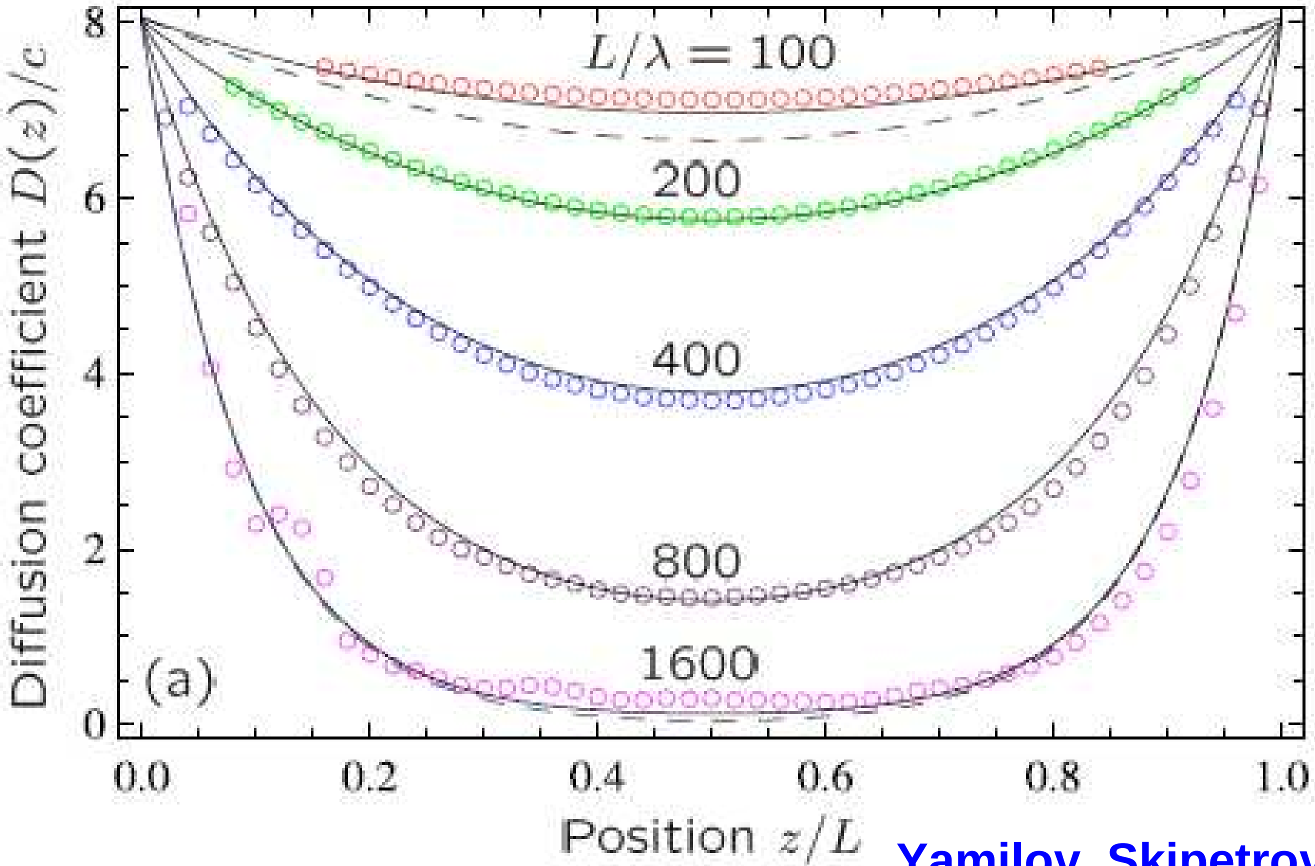
Quasi-discrete modes
Random matrix theory

Courtesy: A.Z. Genack (Queen College, NYC)

Quasi-1D wave guide



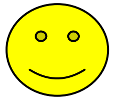
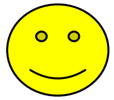
$$D(x) \equiv -\frac{\langle J_x(x) \rangle}{\langle \partial_x W(x) \rangle}$$



Yamilov, Skipetrov, etal 2010

IV. From matter waves to classical waves

IV.a analogies and differences

	electrons	classical waves	
Wave equation	$i\hbar\partial_t\psi = \left(\frac{\hbar^2\nabla^2}{2m} + V(\mathbf{r})\right)\psi$	$\frac{\varepsilon(\mathbf{r})}{c_0^2}\partial_t^2\psi(\mathbf{r},t) - \nabla^2\psi(\mathbf{r},t) = 0$	
Wave	Ψ probability amplitude	Ψ electric field, elastic dispacement,..	
conserved	$\int d\mathbf{r}\, \lvert\psi\rvert^2$ **proba**	$\int d\mathbf{r}\left[\frac{1}{2}\frac{\varepsilon(\mathbf{r})}{c_0^2}\lvert\partial_t\psi\rvert^2 + \frac{1}{2}\lvert\nabla\psi\rvert^2\right]$ **Energy**	
scattering	$V(\mathbf{r}) \Rightarrow \ell(E=0) > 0$	$V(\mathbf{r}) = \frac{(\hbar\omega)^2}{2mc_0^2}[1-\varepsilon(\mathbf{r})]$ $\Rightarrow \ell(\omega\to 0) \propto 1/\omega^{d+1}$	
current	$\mathbf{J} = \frac{\hbar}{m}\mathrm{Im}\,\psi(\nabla\psi)^*$	$\mathbf{J} = \mathrm{Re}\,\partial_t\psi(\nabla\psi)^*$	
detection	All channel in all channel out (conductance)	One channel in one channel out (scattering matrix	
Hilbert space	Direct products →decoherence L_Φ	direct sums → Absorption L_{abs}	

IV.b mesoscopic regime

Mesoscopic regime

$$\ell < L < L_{abs} \text{ or } L_{\varphi}$$

1. Electrons: $\begin{cases} \ell \approx 1\,\text{nm} \\ L_{\phi} \approx 10\,\mu\text{m}\,(1\text{K}) \end{cases}$ **NANO**

   Cold atoms $\begin{cases} \ell \approx 300\ \text{nm} \\ L_{\phi} > 1\,mm \end{cases}$ **MICRO**

2. Photons $\begin{cases} \ell \approx 300\,\text{nm} - 1\,\text{mm} \\ L_a \approx 100\,\mu\text{m-1cm} \end{cases}$ **MICRO-MILLI**

3. Micro waves $\begin{cases} \ell \approx 5\,\text{cm} \\ L_a \approx 50\,\text{cm} \end{cases}$ **CENTI**

4. Seismic waves $\begin{cases} \ell \approx 100\ \text{km} \\ L_a \approx 150\ \text{km}(1\text{Hz}) \end{cases}$ **KILO**

IV.c Energy velocity of classical waves

$$V(r,\omega) \Rightarrow \quad \Sigma(\omega+\tfrac{1}{2}\Omega,\mathbf{k}+\tfrac{1}{2}\mathbf{q})-\Sigma^*(\omega-\tfrac{1}{2}\Omega,\mathbf{k}-\tfrac{1}{2}\mathbf{q})$$

$$=\sum_{\mathbf{k}''} U_{\omega\mathbf{k}\mathbf{k}''}(\Omega,\mathbf{q})+\delta(\omega)i\Omega$$

$$\frac{\mathrm{d}}{\mathrm{d}E}=\left(\frac{\partial}{\partial E}\right)_V+\frac{\mathrm{d}V_0}{\mathrm{d}E}\left(\frac{\partial}{\partial V_0}\right)_E$$

Dwell time

$$-i\Omega[1+\delta(\omega)]\sum_{\mathbf{k}}\Phi_{E\mathbf{k}\mathbf{k}'}(\Omega,\mathbf{q})+i\mathbf{q}\cdot\sum_{\mathbf{k}}\mathbf{k}\,\Phi_{E\mathbf{k}\mathbf{k}'}(\Omega,\mathbf{q})=A(E,k')$$

$\mathbf{E}^2$ radiation

$\mathbf{E}\cdot\mathbf{P}$ matter

Diffusion approximation for classical waves

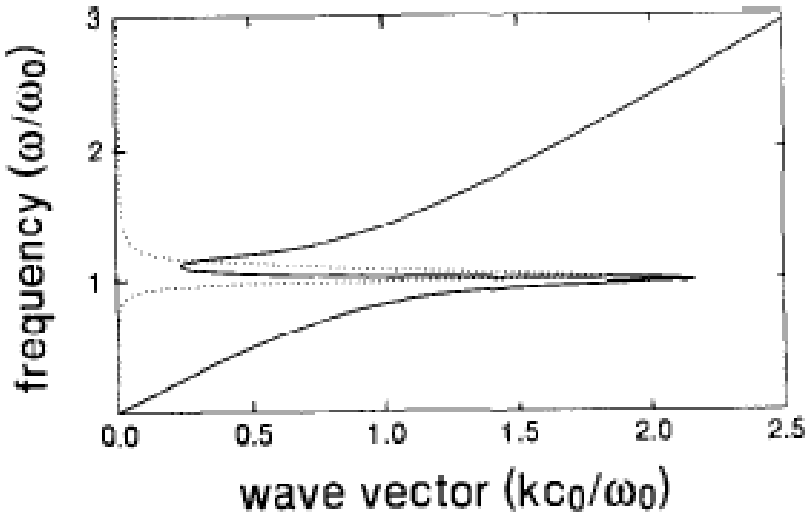
$$\Phi_{\omega \mathbf{k}\mathbf{k}'}(\Omega\ ,\mathbf{q}) = \frac{2\pi}{\rho(\omega)} \frac{\phi(\omega,\mathbf{k},\mathbf{q})\phi(\omega,\mathbf{k}',\mathbf{q})}{-i\Omega(1+\delta) + \frac{1}{d}\frac{c_0^2}{v_p}\ell^* q^2}$$

$\frac{c_0^2}{v_p} \rightarrow < v >$

$$= \frac{2\pi}{\rho(\omega)(1+\delta)} \frac{\phi(\omega,\mathbf{k},\mathbf{q})\phi(\omega,\mathbf{k}',\mathbf{q})}{-i\Omega + Dq^2}$$

DOS of radiation + matter

$$D(E) = \frac{1}{3}\frac{c_0^2}{v_p}\frac{1}{1+\delta(\omega)}\ell^*(E) \Rightarrow \mathrm{v}_E = \frac{c_0^2}{v_p}\frac{1}{1+\delta(\omega)}$$



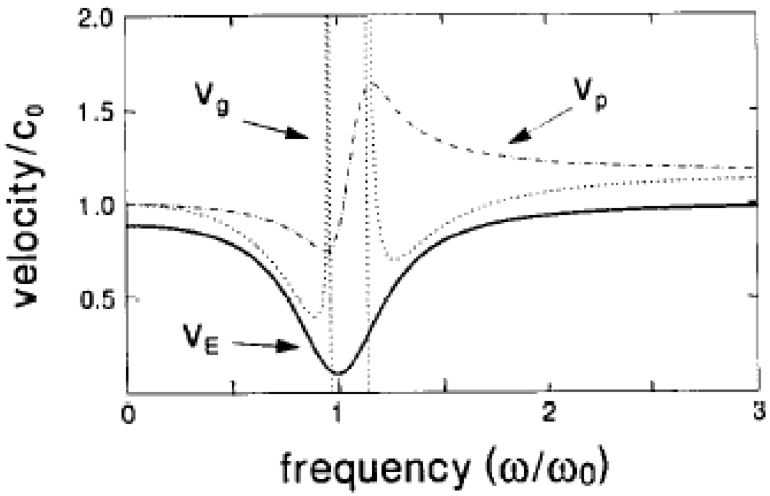
$$n(\omega) = 1 + \rho\alpha(\omega)$$

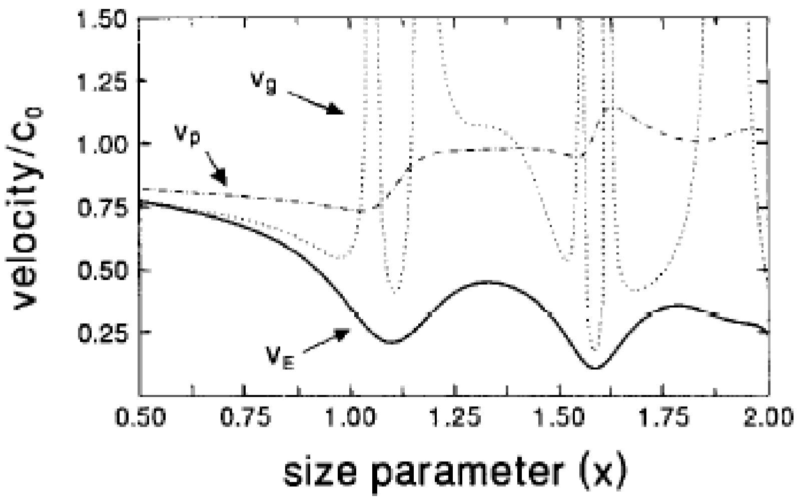
$$k = n(\omega)\omega / c_0$$

Mie spheres

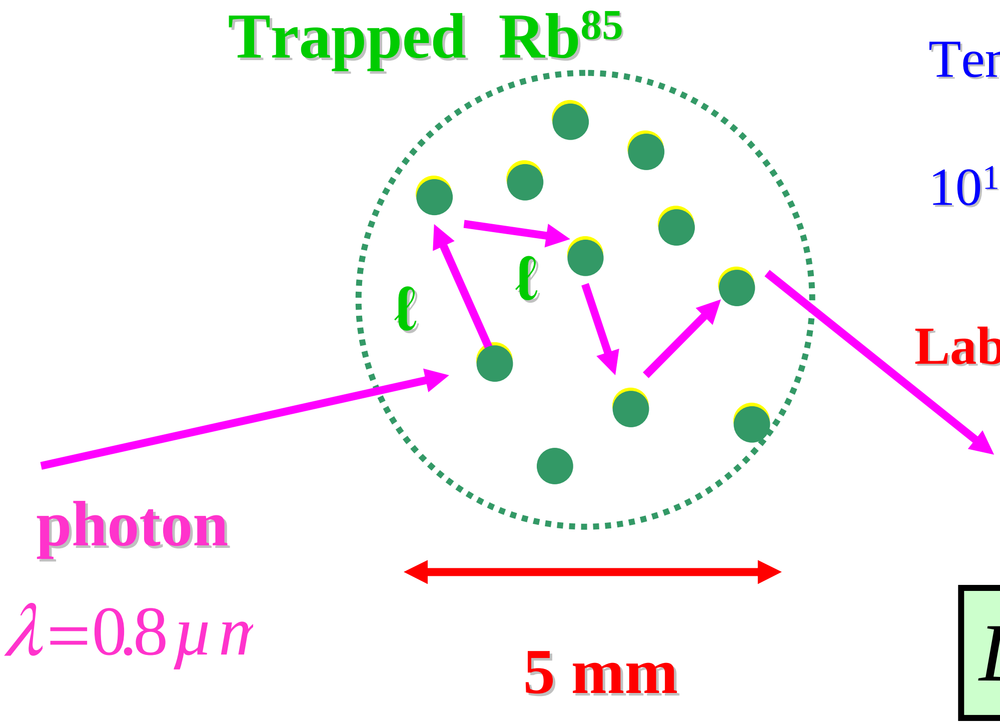
Harmonic oscillators

$$\frac{v_E}{c_0} = \frac{c_0^2}{c_p^2}\left[1 + \frac{3}{4}\frac{f}{x^2}\sum_{n=1}^{\infty}(2n+1)\left(\frac{d\alpha_n}{dx} + \frac{d\beta_n}{dx}\right)\right]^{-}$$





Small diffusion constant ≠ localization !



Temperature T = 0,0001 K (v=15 cm/s)

10^{10} atomes

Labeyrie, Miniatura, Kaiser (2005, Nice)

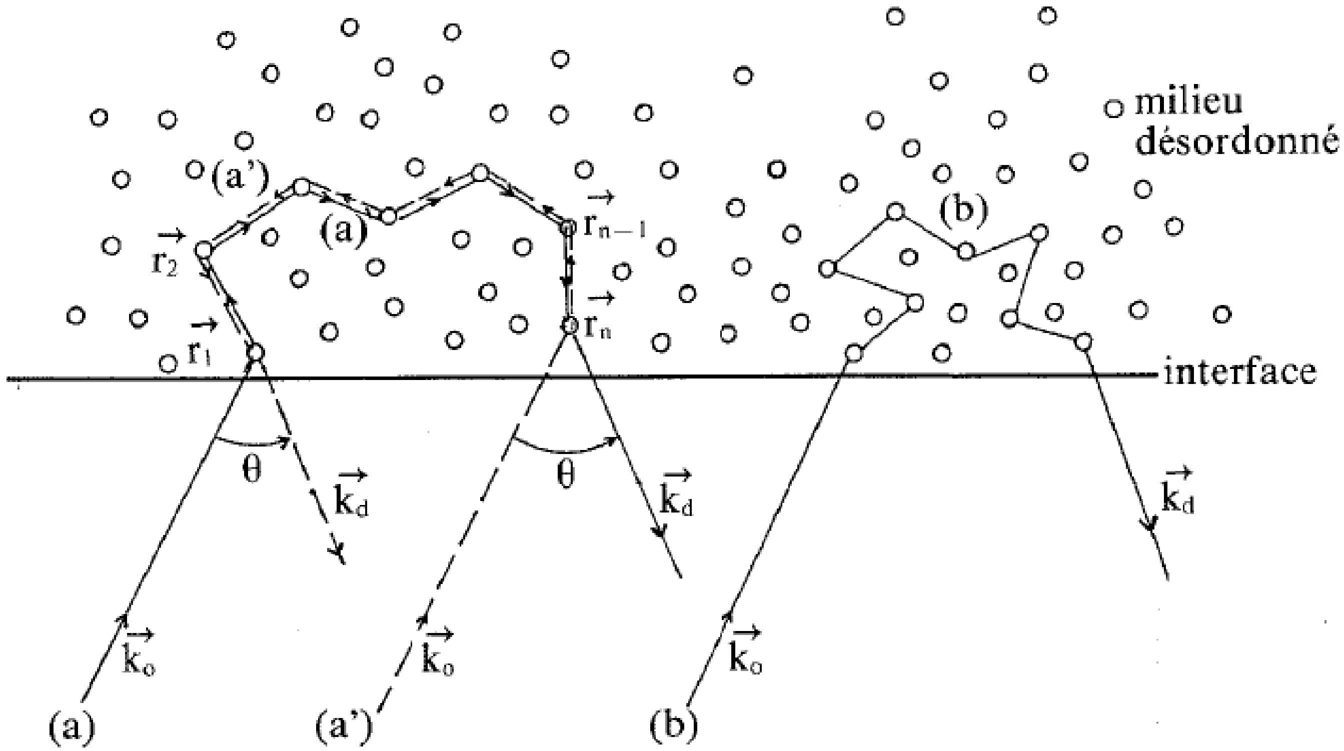
$$D = \tfrac{1}{3} v_E \ell = 0.9\, m^2 / \mathrm{sec}$$

Random walk of photons

$$v_E = \ldots\ldots\ldots \quad 0.00003\ c_0 \qquad \delta = 10^5$$

$$\ell = 0.3 mm \quad k\ell = 4000$$

V. Enhanced Backscattering as a precursor of localization



Akkermans, Maynard, Wolf, Maret,
Van Albada, Lagendijk
1986

V. a Reciprocity

$$T_{i\mathbf{k}j\mathbf{k}'} = T_{j-\mathbf{k}'i-\mathbf{k}}$$

$$\exp(i\mathbf{k}\cdot\mathbf{r}_1)T_{i\mathbf{k}i_1\mathbf{k}_1}\exp[i\mathbf{k}_1\cdot(\mathbf{r}_2-\mathbf{r}_1)]T_{i_1\mathbf{k}_1i_2\mathbf{k}_2}\cdots\exp[i\mathbf{k}_{n-1}\cdot(\mathbf{r}_n-\mathbf{r}_{n-1})]T_{i_{n-1}\mathbf{k}_{n-1}j\mathbf{k}'}\exp(-i\mathbf{k}'\cdot\mathbf{r}_n)$$

$$=$$

$$\exp(-i\mathbf{k}'\cdot\mathbf{r}_n)T_{j-\mathbf{k}'i_{n-1}-\mathbf{k}_{n-1}}\exp[-i\mathbf{k}_{n-1}\cdot(\mathbf{r}_{n-1}-\mathbf{r}_n)]\cdots T_{i_2-\mathbf{k}_2i-\mathbf{k}}\exp[-i\mathbf{k}_1\cdot(\mathbf{r}_1-\mathbf{r}_2)]T_{i_1-\mathbf{k}_1i-\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{r}_1)$$

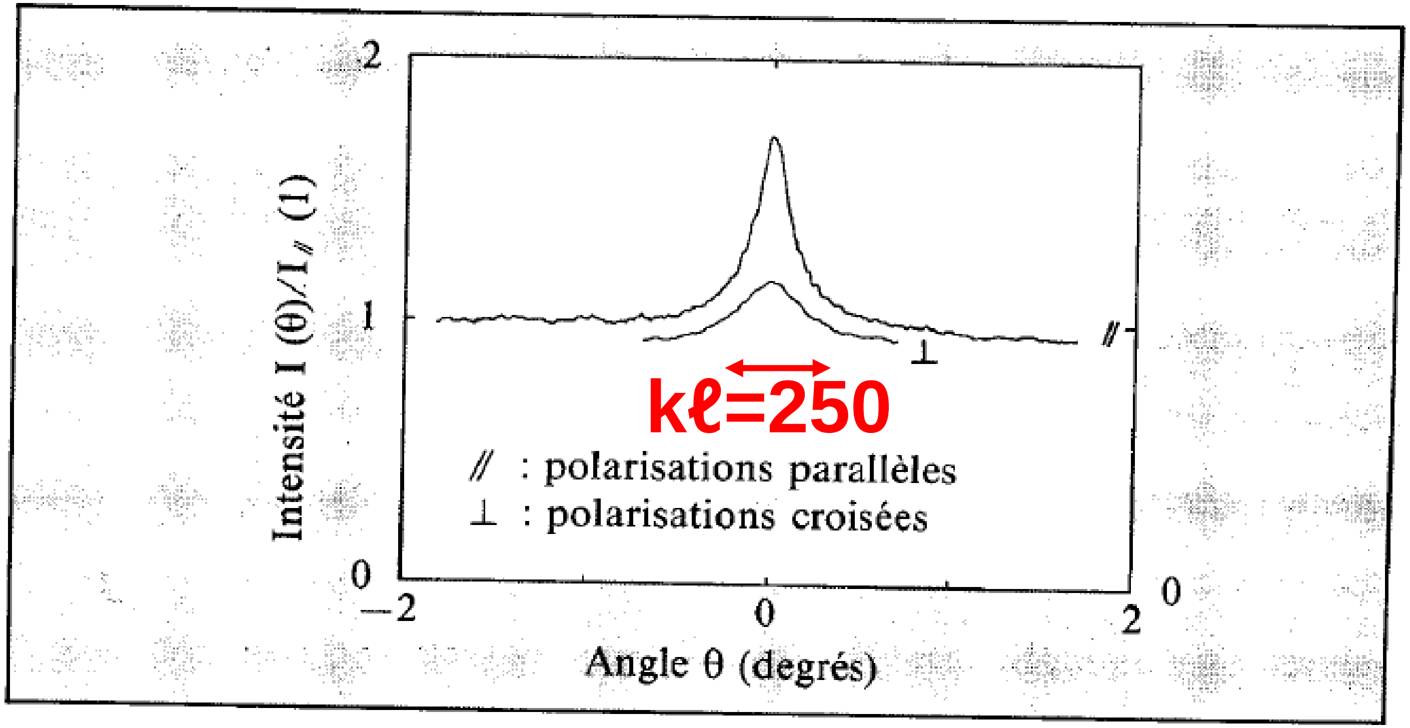
$$\Psi_{i\mathbf{k}\to j\mathbf{k}'}(1\to 2\to\cdots n) = \Psi_{j-\mathbf{k}'\to i-\mathbf{k}}(n\to\cdots 2\to 1)$$

$$\Psi_{i\mathbf{k}\to i-\mathbf{k}}(1\to 2\to\cdots n) = \Psi_{i\mathbf{k}\to i-\mathbf{k}}(n\to\cdots 2\to 1)$$

$$\left|\Psi(A\to B)+\Psi(B\to A)\right|^2 = \left|\Psi(A\to B)\right|^2 + \left|\Psi(B\to A)\right|^2 + 2\,\mathrm{Re}\,\Psi(A\to B)\Psi^*(B\to A)$$

Constructive interference of oppositely propagating paths at backscattering in equal polarization channels → Factor two enhancement at backscattering

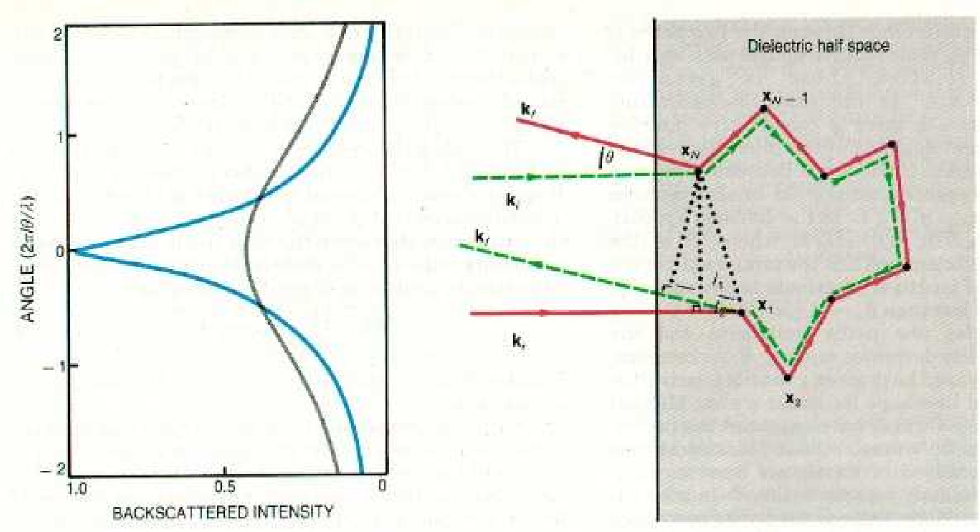
Enhanced Backscattering as a precursor



0.46 µm Mie spheres, 10 vol-%, wavelength 513 nm

(Wolf, Maret, 1986)

Enhanced Backscattering as a precursor



$$Q_0(\mathbf{r}-\mathbf{r}') = [4\pi D|\mathbf{r}-\mathbf{r}'|]^{-1}$$

halfspace $\longrightarrow$ $Q(\mathbf{r}-\mathbf{r}') = Q_0(\mathbf{r}-\mathbf{r}') - Q_0(\mathbf{r}-\mathbf{r}'^{*})$

$$\alpha(\theta) \propto \int d^2\boldsymbol{\rho}[1+\cos\mathbf{q}\cdot\boldsymbol{\rho}]\, Q(z=\ell, z'=\ell, \rho)$$

$$\alpha(\theta) = \frac{3}{8\pi}\left[1+\frac{2z_0}{l}+\frac{1}{(1+q_\perp l)^2}\left(1+\frac{1-\exp(-2q_\perp z_0)}{q_\perp l}\right)\right]$$

$q_\perp = 2k \sin \vartheta/2$ $\quad$ $z0 = 2/3\ \ell^*$

(Akkermans, Maynard, 1986)

Boundary condition for diffuse propagator

$$-D\nabla^2 Q(\mathbf{r}) = \delta(\mathbf{r})$$

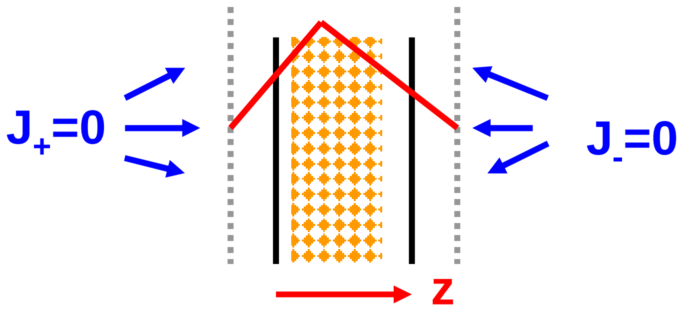
$$\phi(E,\mathbf{k},\mathbf{r}) = \frac{2\pi}{\rho(\omega)[1+\delta(\omega)]}[A(\omega,k) - \mathbf{k}\cdot\nabla F(\omega,k)]Q(\mathbf{r})$$

Flux to the right:

$$J_+ = c_0 \sum_{\mathbf{k}\cdot\hat{\mathbf{z}}>0} \hat{\mathbf{k}}\cdot\hat{\mathbf{z}}\phi(\omega,\mathbf{k},\mathbf{r}) = \left(\frac{\mathrm{v}_E}{4} - \frac{1}{2}\times\frac{1}{3}\mathrm{v}_E \ell^* \partial_z\right)Q(\mathbf{r}) \approx \frac{\mathrm{v}_E}{4}Q(z - \tfrac{2}{3}\ell^*, \boldsymbol{\rho})$$

Flux to the left:

$$J_- = c_0 \sum_{\mathbf{k}\cdot\hat{\mathbf{z}}<0} -\hat{\mathbf{k}}\cdot\hat{\mathbf{z}}\phi(\omega,\mathbf{k},\mathbf{r}) \approx \frac{\mathrm{v}_E}{4}Q(z + \tfrac{2}{3}\ell^*, \boldsymbol{\rho})$$

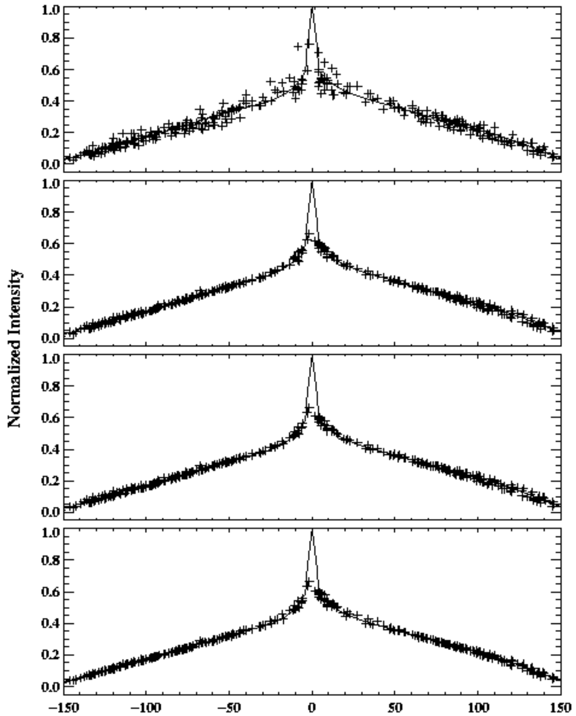


$$Q(-\tfrac{2}{3}\ell^*) = 0$$

$$Q(L + \tfrac{2}{3}\ell^*) = 0$$

Coherent Backscattering in natural media

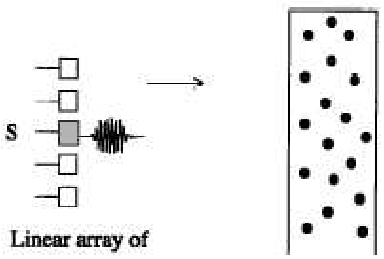
Opposition effect from Saturn's rings: CBS? (M. Mishchenko)

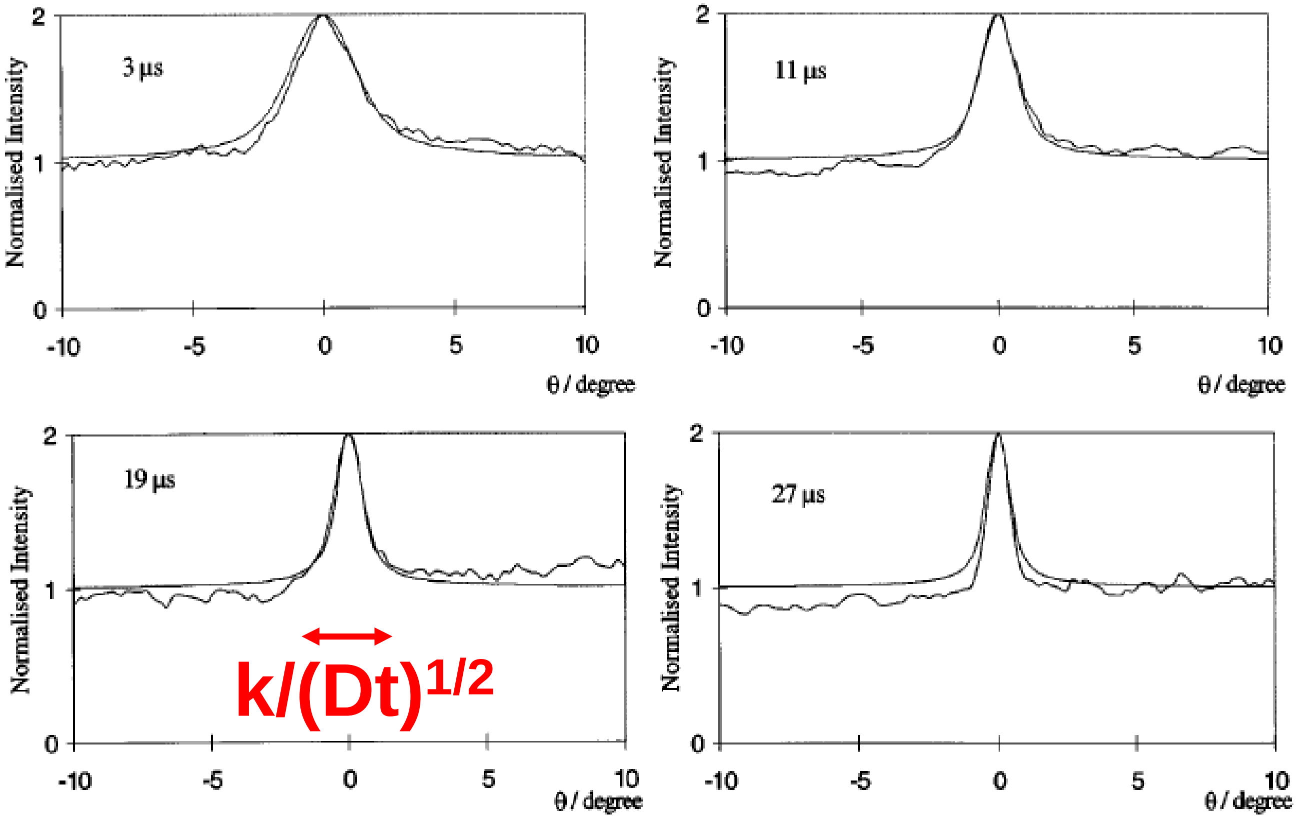


Phase angle

Enhanced Backscattering of ultrasound

Derode, Fink,
n Tiggelen, 1997





$$I(x,t) = 1 + \frac{1}{b} \times \int_{-\pi/2}^{\pi/2} g(\theta)\exp\left(-\frac{Dt}{a^2}(kx\cos\theta)^2\cos^4\theta\right)d\theta$$

$$g(\theta) = \cos^2\theta(\pi/4 + \cos\theta)^2$$

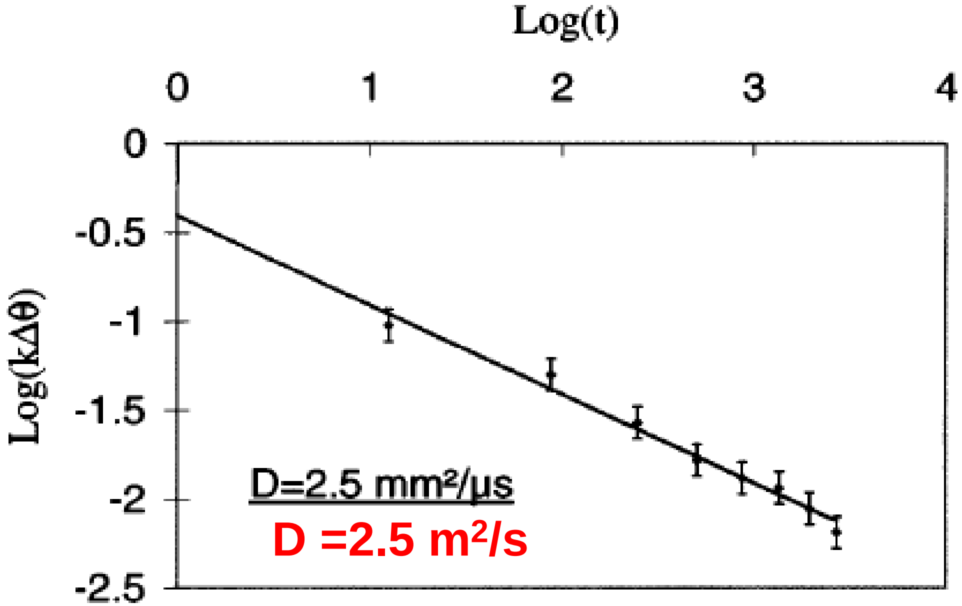


FIG. 4. Cone width versus time. Slope of the fit: 0.49.