Anderson Localization of Light and Ultrasound

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laboratoire de physique et de modélisation des milieux condensés

dépasser les frontières

Université Joseph Fourier – Grenoble 1 / CNRS

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   - Historical perspective
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   - Supersymmetric theory
   - Selfconsistent theory of localization

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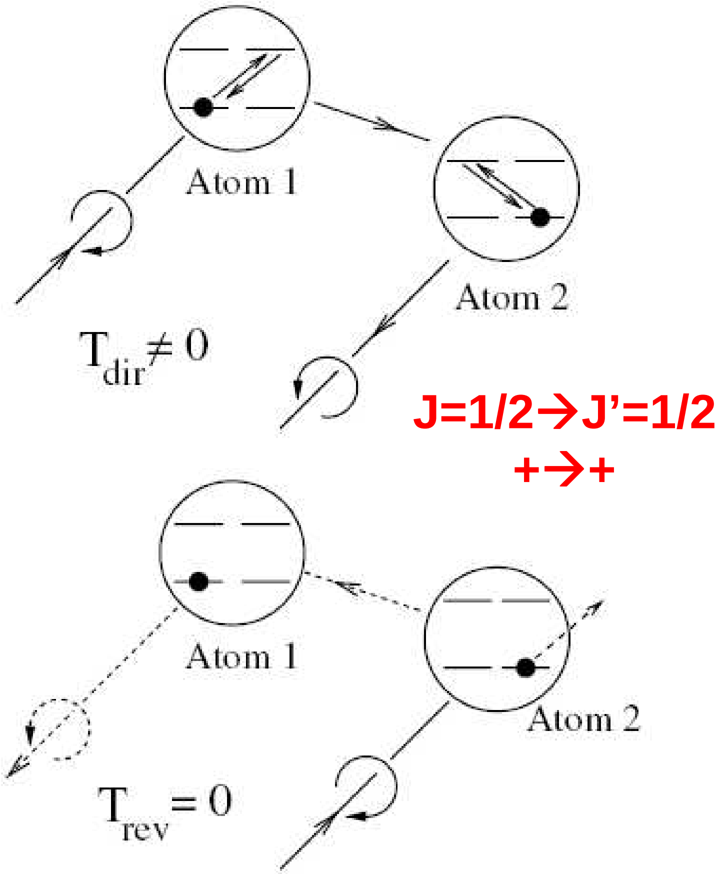
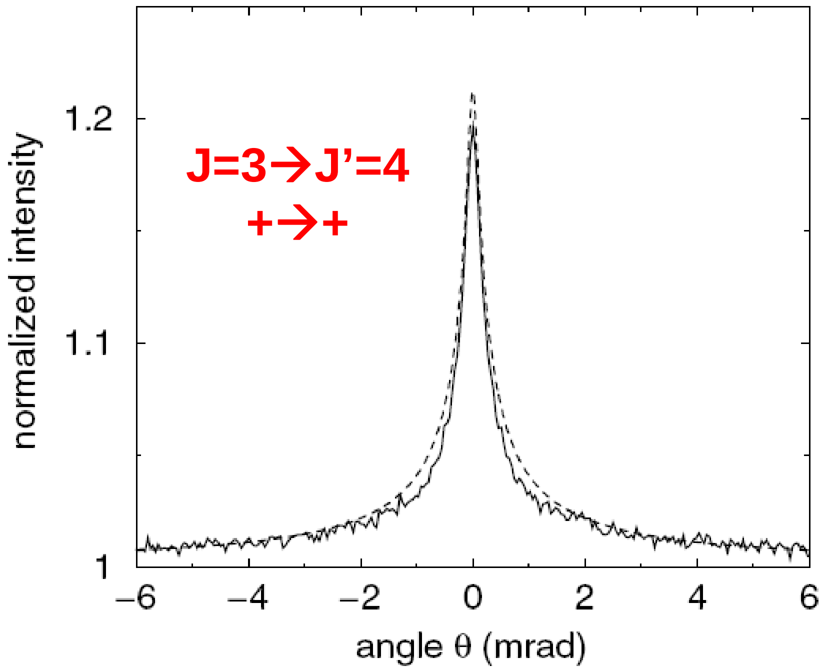
Contents of the course

5. **Anderson localization of noninteracting atoms in 3D**
   - Self-consistent Born approximation in speckle potential
   - Phase diagram from selfconsistent theory
   - Energy distribution of atoms

Coherent Backscattering of Light by Cold Atoms

G. Labeyrie, F. de Tomasi,* J.-C. Bernard, C. A. Müller, C. Miniatura, and R. Kaiser
Institut Non Linéaire de Nice, UMR 6618, 1361 route des Lucioles, F-06560 Valbonne, France
(Received 30 July 1999)



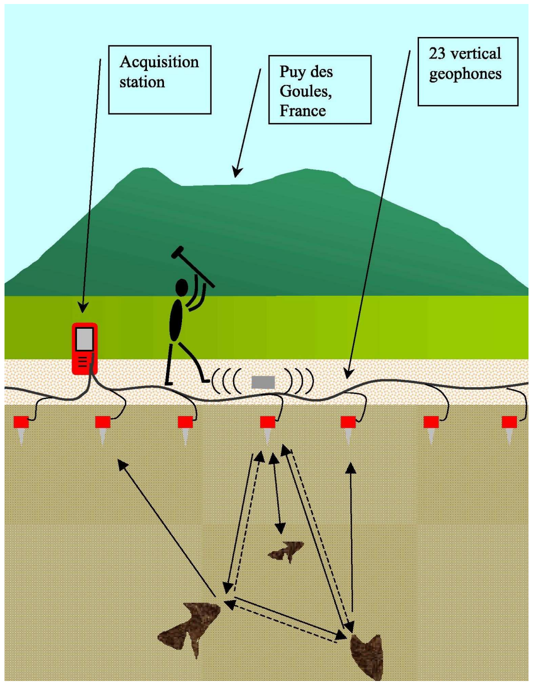


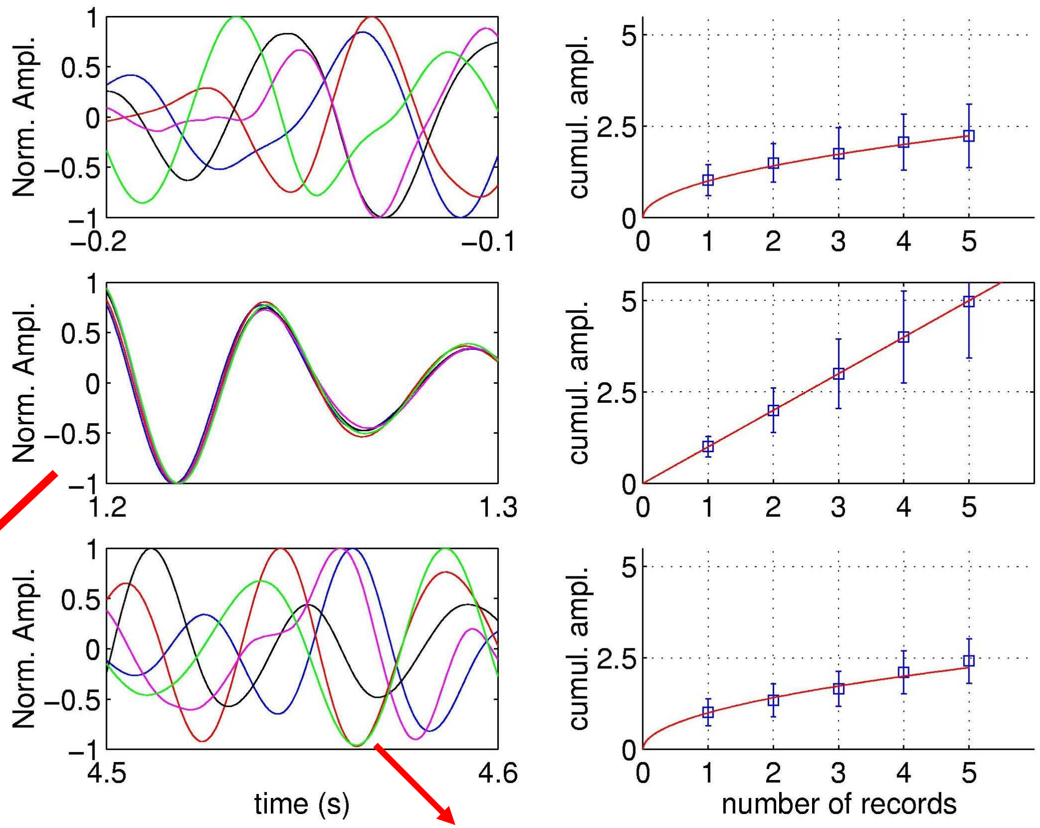
$$T_{\rm dir}(\mathbf{k}\boldsymbol{\epsilon},\{m\}\rightarrow\mathbf{k}'\boldsymbol{\epsilon}',\{m'\}) = (-1)^{\sum_i(m_i'-m_i)}T_{\rm rev}(-\mathbf{k}'\boldsymbol{\epsilon}'^*,-\{m'\}\rightarrow -\mathbf{k}\boldsymbol{\epsilon}^*,-\{m\})\,.$$

Coherent backscattering with seismic waves in Auvergne

Larose etal, PRL 2004

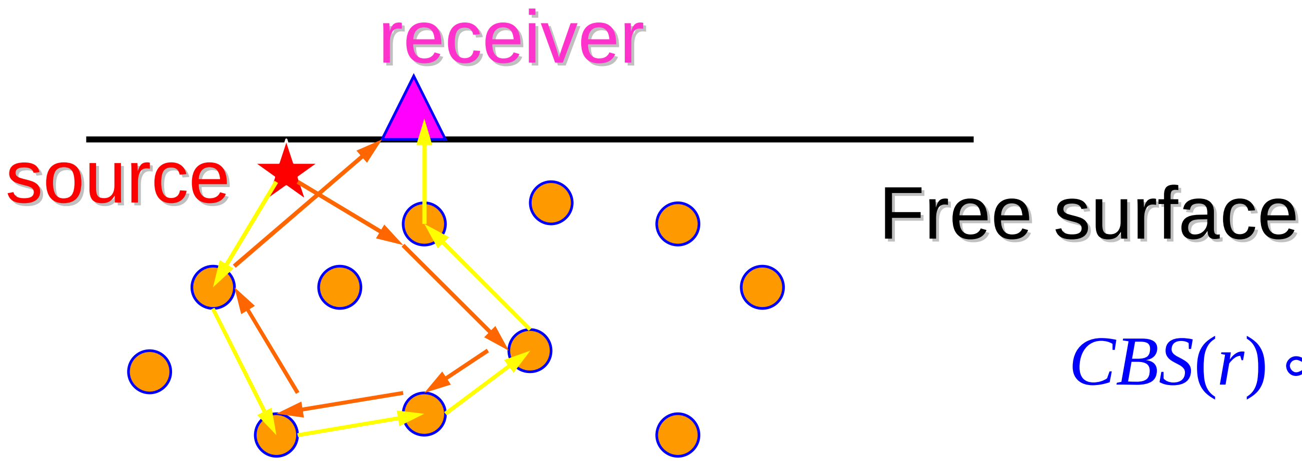
Operator noise





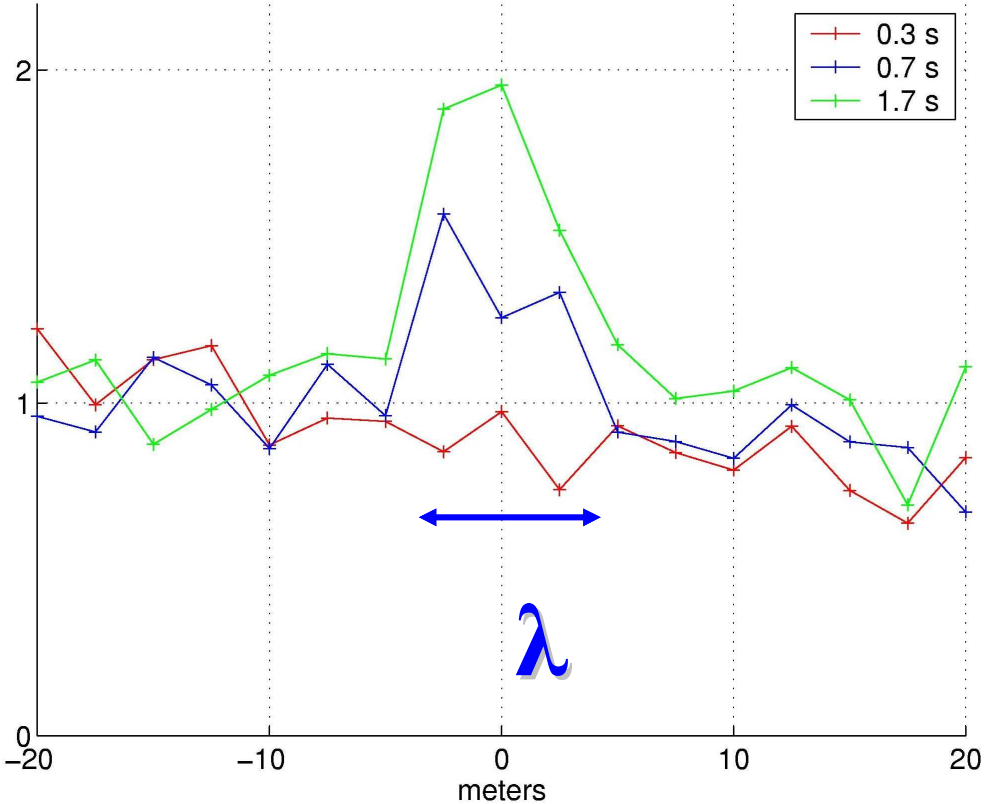
Mesoscopic signal

Background noise



$$CBS(r) \propto 1 + J_0^2\left(\frac{2\pi r}{\lambda}\right)$$

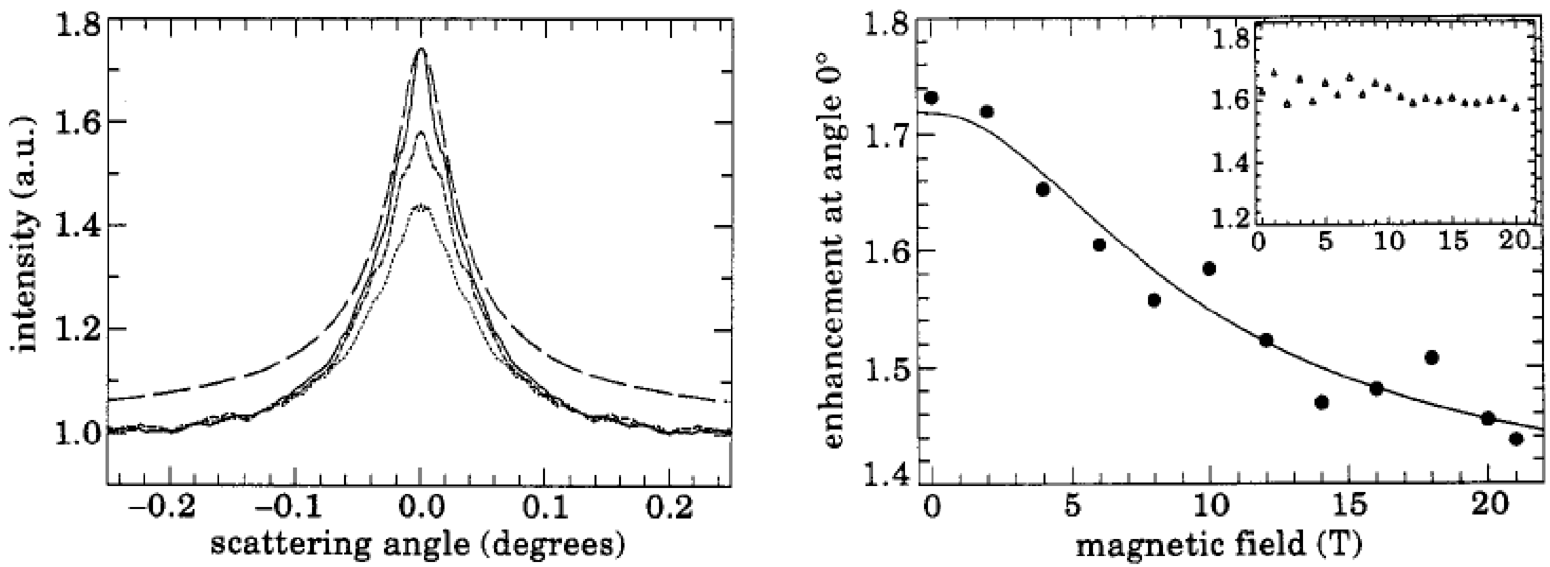
Enhancement factor



Enhanced Backscattering in a magnetic field

$$\varepsilon_{nm} = m^2 \delta_{nm} + \frac{iV}{k} \varepsilon_{ijk} B_k$$

Faraday effect

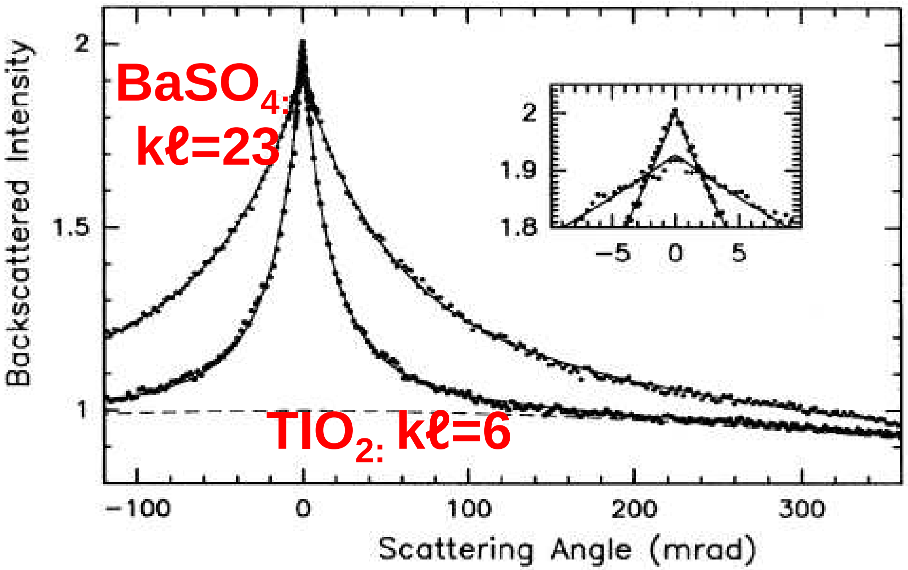


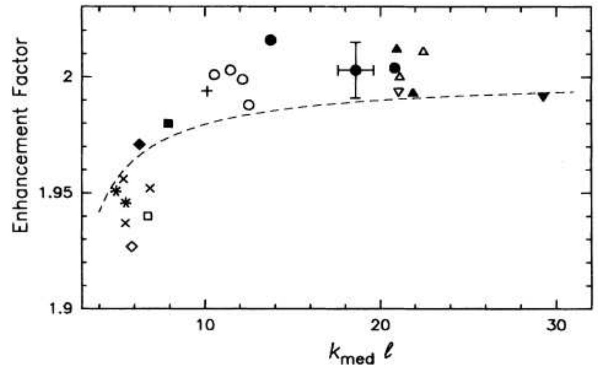
$$T_{i\mathbf{k}j\mathbf{k}'}(\mathbf{B}) = T_{j-\mathbf{k}'i-\mathbf{k}}(-\mathbf{B})$$

$$\Psi_{i\mathbf{k}\to i-\mathbf{k}}(1 \to 2 \to \cdots n, \mathbf{B}) = \Psi_{i\mathbf{k}\to i-\mathbf{k}}(n \to \cdots 2 \to 1, -\mathbf{B})$$

Erbacher, Lenke and Maret, EPL 1993

Enhanced Backscattering in strongly scattering samples





→kℓ

Wiersma etal 1995

$$\Psi_{i\mathbf{k}\to j\mathbf{k}'}(1\to 2\to\cdots 1)=\Psi_{j-\mathbf{k}'\to i-\mathbf{k}}(1\to\cdots 2\to 1)=L_n\exp\left[i\mathbf{r}_1\cdot(\mathbf{k}-\mathbf{k}')\right]$$

$$\left|\Psi(1\to n)+\Psi(n\to 1)+\Psi(1\to 1)+\Psi(n\to n)\right|^2$$

$$=B+C\left(\theta\right)+L$$

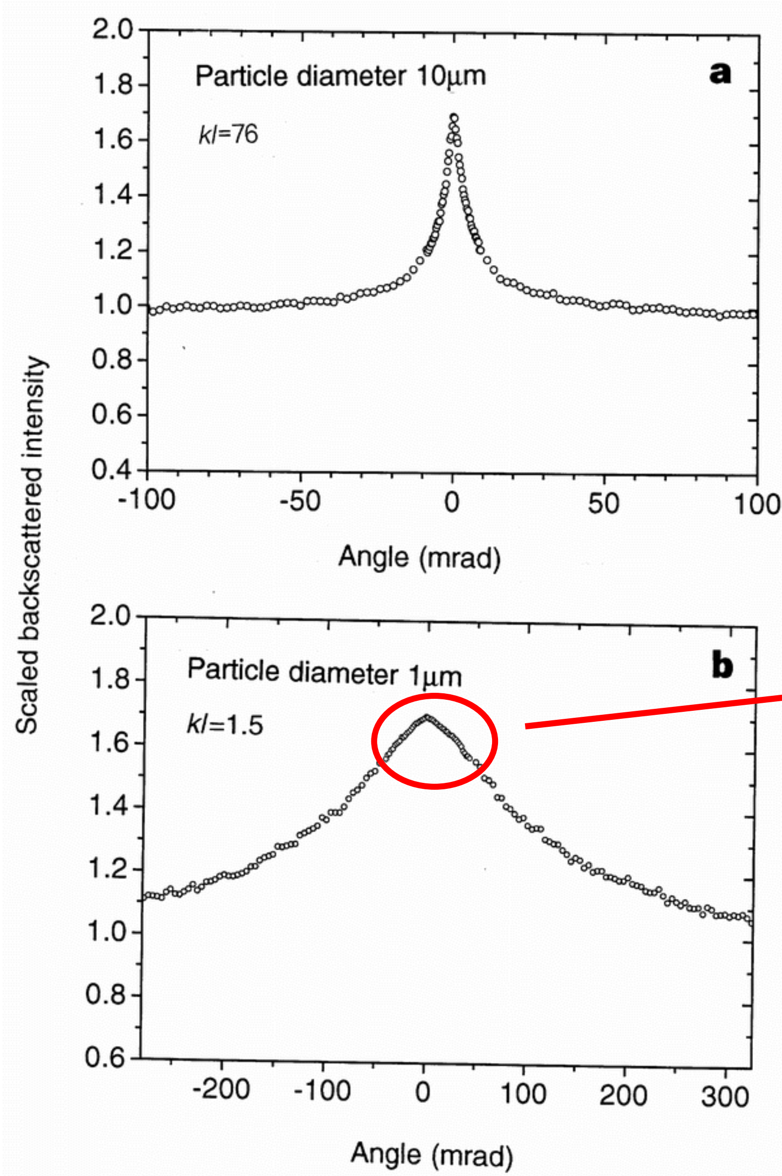
Incoherent background

CBS

Coherent background

Peak/background ≠ 2

Enhanced Backscattering in strongly scattering samples



Suppressed long paths?

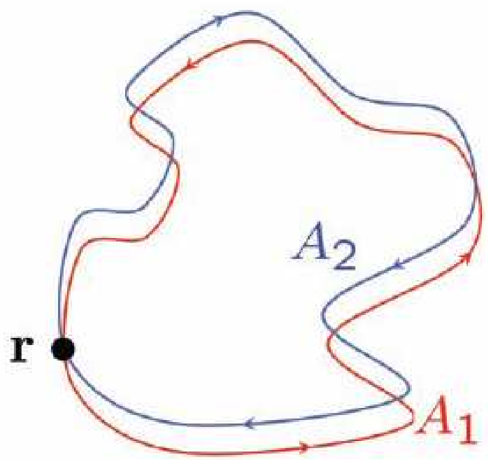
Localization looks like absorption?

Wiersma etal, Nature 1997

V.c How much extra energy returns at to the source by constructive interference?

$$\text{interference energy}/\text{Source} = \mathrm{v_E}\int_{\ell/\mathrm{v}}^{\infty} dt \left(\frac{\lambda}{2\pi}\right)^{d-1} \frac{1}{(Dt)^{d/2}}$$

wave volume at source x **Probability density of quantum diffusion at source**



$P_{\text{particule}} = |A_1|^2 + |A_2|^2$

$P_{\text{onde}} = |A_1 + A_2|^2$

$P_{\text{onde}} = 2P_{\text{particule}}$

$$= \quad 3D: \frac{1}{(k\ell)^2}$$

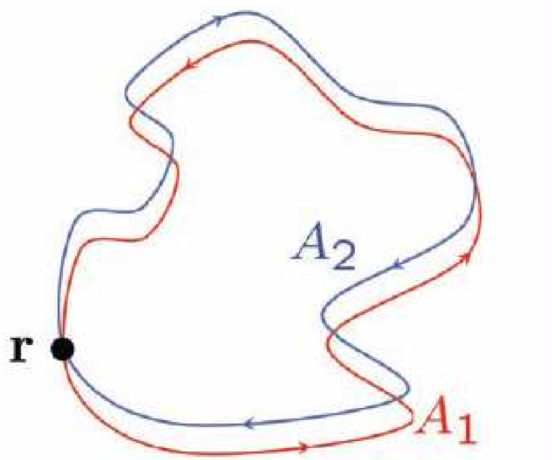
$$2D \& 1D: \infty$$

Something is wrong with diffusion picture at large times

Enhanced backscattering is precursor

…..and in *open* quasi 1D wave guide ?

$$\text{interference energy} \Big/ \text{Source} = v_E \int_0^{L^2/D} dt \left(\frac{\lambda}{2\pi}\right)^2 \frac{1}{A} \frac{1}{(Dt)^{1/2}}$$



$P_{\text{particule}} = |A_1|^2 + |A_2|^2$

$P_{\text{onde}} = |A_1 + A_2|^2$

$P_{\text{onde}} = 2P_{\text{particule}}$

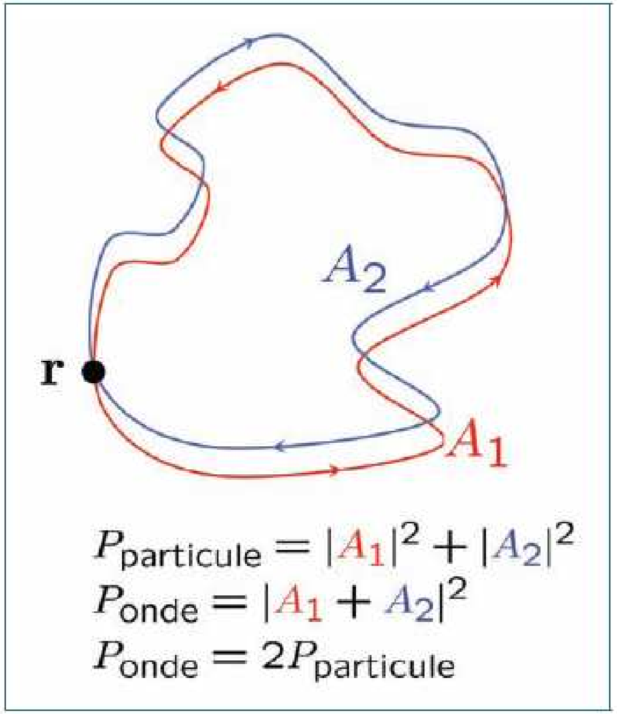
$$= \frac{1}{Ak^2}\frac{\ell}{L} \equiv \frac{1}{g}$$

Number of transverse channels N

Something is wrong with diffusion picture for g < 1 (L > Nℓ)

…..and in *open* d-dimensional media ?

$$\text{interference energy}\big/\text{Source} = v_E \int_0^{L^2/D} dt \left(\frac{\lambda}{2\pi}\right)^{d-1} \frac{1}{L^d}$$



$$= \quad \frac{v_E}{k^{d-1} D L^{d-2}}$$

- **d=2 critical dimension for localization**
- **velocity cancels**
- **d>2: Set of points where this equals 1:**

$$D \propto v_E \frac{\ell^{d-1}}{L^{d-2}} \qquad k\ell \approx 1$$

A critical point in d>2 near kℓ=1 with scale dependent diffusion?

Localization in *open* media ?

$$N(E) \approx L^d \int \frac{d^d\mathbf{k}}{(2\pi)^d}\,\delta\!\left(E - \frac{\hbar^2 k^2}{2m}\right)$$

Thouless number

(Quantum return Probability) $^{-1}$

$$\frac{k_F{}^{d-1} D L^{d-2}}{v_F} = \frac{k_F{}^{d-1} L^d}{\hbar v_F}\hbar D / L^2 = \begin{cases} N(E)\,\delta E_{\text{Thouless}} = \dfrac{\delta E_{\text{Thouless}}}{\Delta E_{\text{level}}} \\[2ex] \rho(E) D(E) \dfrac{L^{d-1}}{L} = \dfrac{G}{e^2/h} \end{cases}$$

dimensionless conductance g

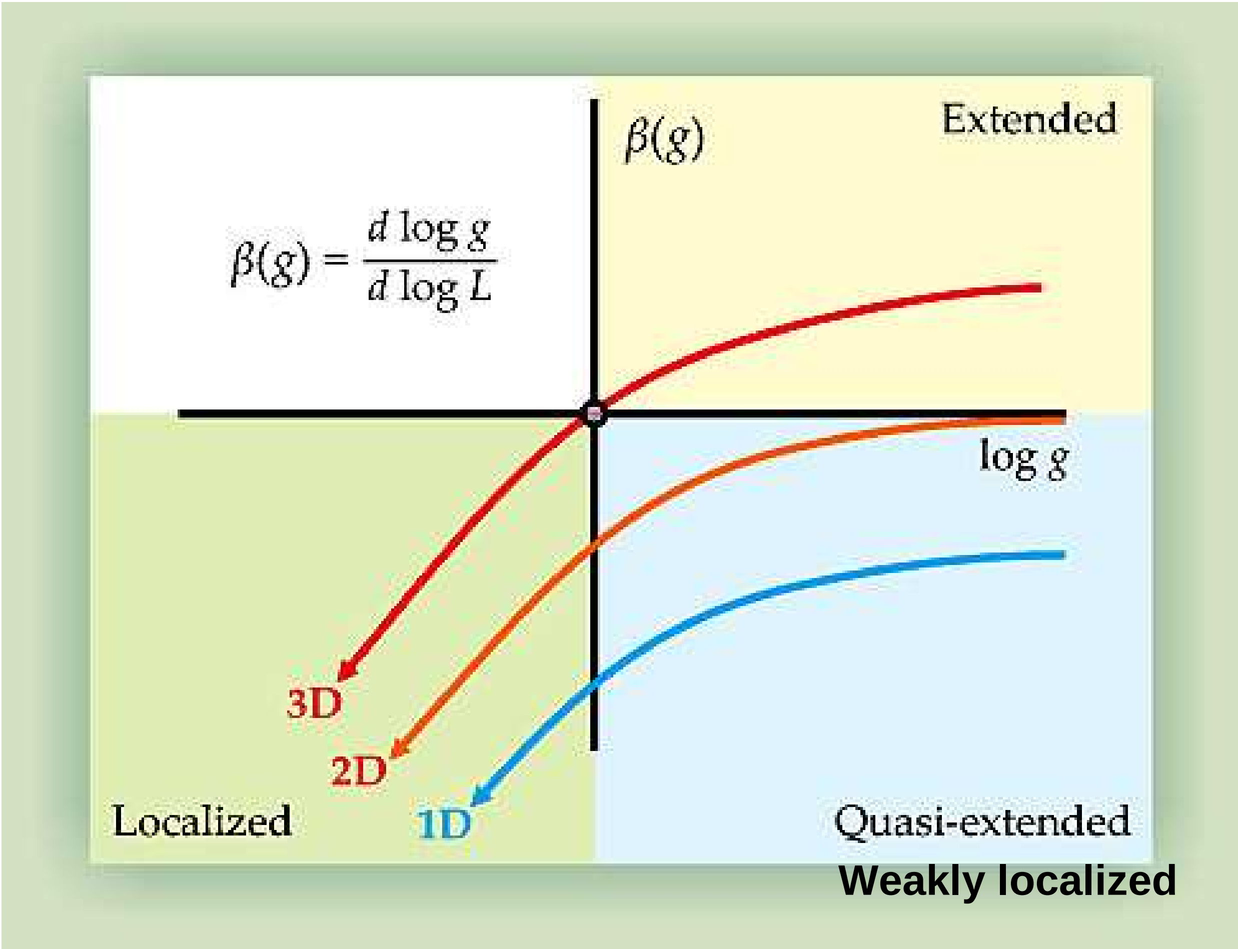
Einstein relation

Onset of Anderson localization

return proba = 1 ⟺ level width = level spacing ⟺ g=1

Thouless, Edwards, 1972

Scaling theory of localization



« Gang of four », 1980

V. Enhanced Backscattering as a precursor of localization

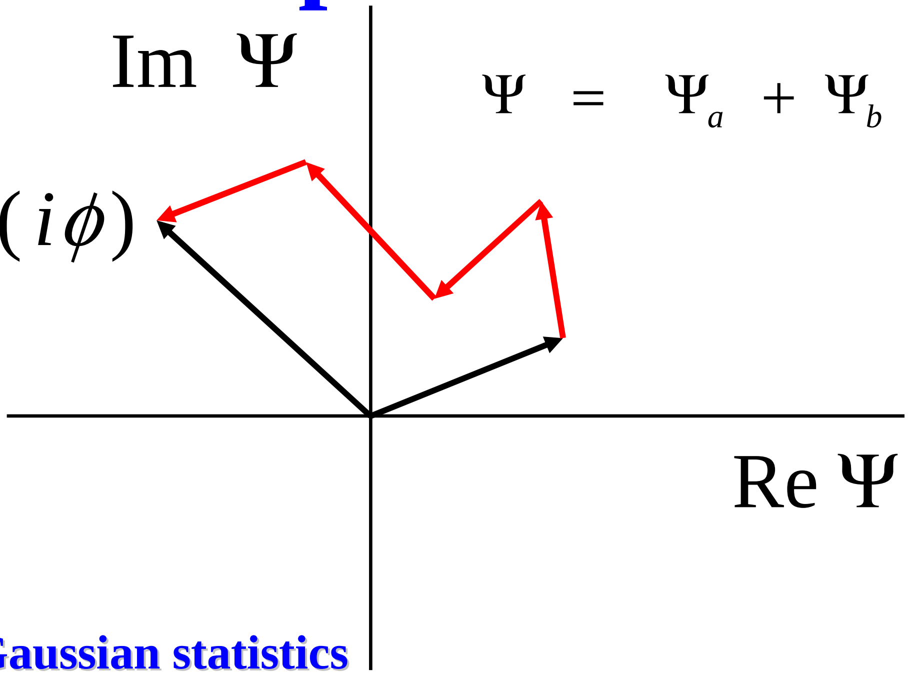
- conclusion: near g=1 constructive interferences dominate wave transport

VI. Speckles and correlations

VI.a Gaussian Statistics of complex field

$$\Psi = E\exp(i\phi)$$

$$\Psi = \Psi_a + \Psi_b + \Psi_c + \ldots$$



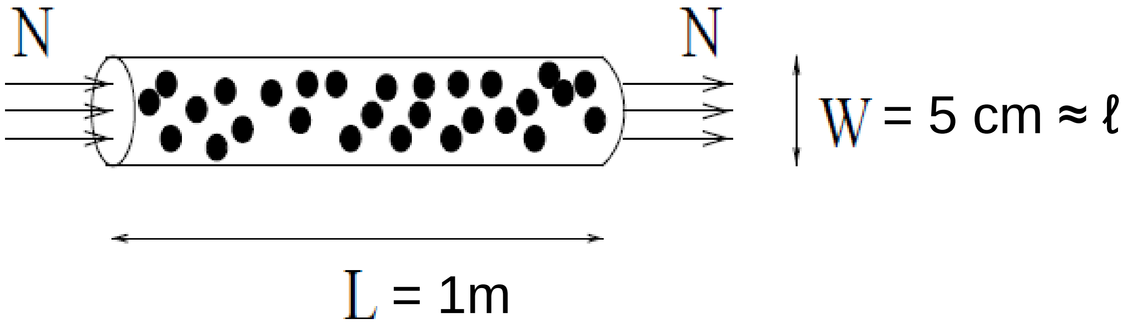
Multiple scattering → Gaussian statistics

$$P(\Psi_1, \Psi_2, \ldots \Psi_N) = \frac{1}{\pi^N \det \mathbf{C}} \exp\left(-\boldsymbol{\Psi}^* \cdot \mathbf{C}^{-1} \cdot \boldsymbol{\Psi}\right)$$

$$C_{ij} \equiv \langle \Psi_i \Psi_j^* \rangle$$

From diffusion equation

Speckles of Micro-waves in Quasi 1D media

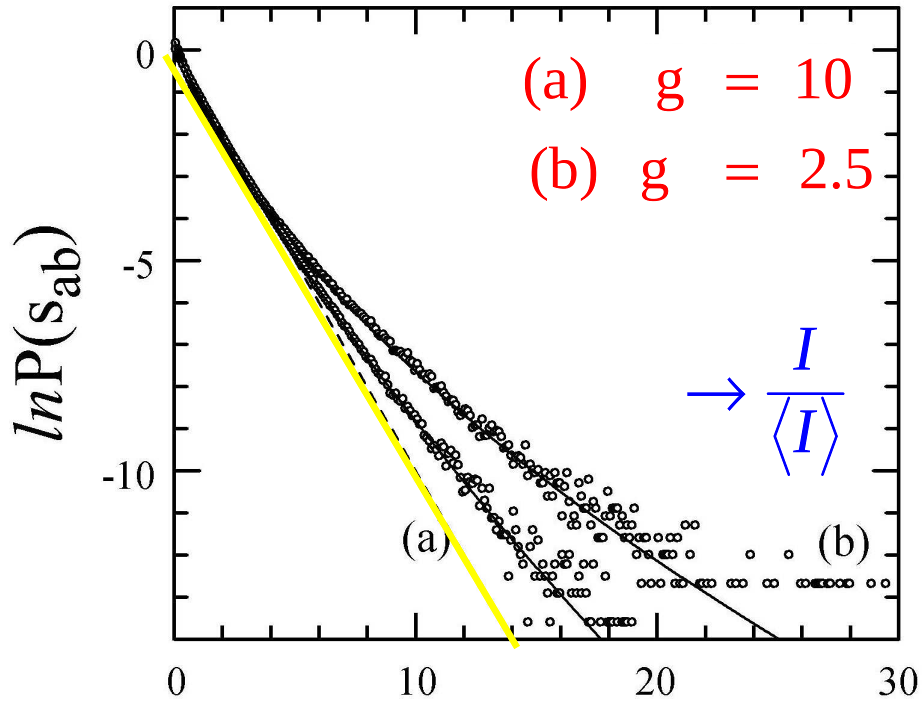


$$t_{ab} = \sqrt{I}\, e^{i\phi}$$

N

$$g = \frac{\zeta}{L} = \frac{Ak^2}{3\pi}\frac{\ell}{L}$$

$P(I)$ ↑



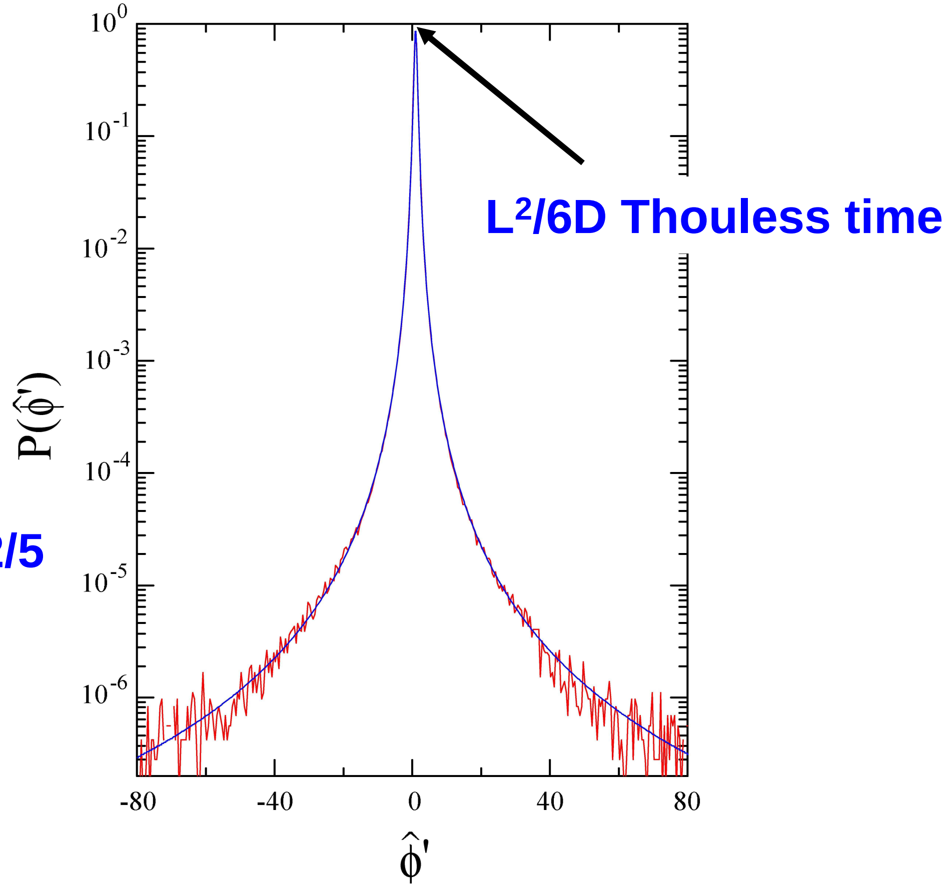
Genack etal, 1990

Phase statistics in diffuse regime

$$P(E_1,\ldots,E_K) = \frac{1}{\pi^K \det \mathbf{C}} \exp\left(-\sum^{K} \bar{E}_i C_{ij}^{-1} E_j\right)$$

$$\longrightarrow \quad P\left(\hat{\phi}' \equiv \frac{\phi'}{\langle \phi' \rangle}\right) = \frac{1}{2} \frac{Q}{[Q + (\hat{\phi}' - 1)^2]^{3/2}};$$

$$P\left(\frac{d\phi}{d\omega}\right)$$



Diffusion approximation → Q=2/5

Genack etal, 1998

VI.b Correlations of intensity

$$\langle I_1 I_2 \rangle = \langle \psi_1 \psi_1^* \psi_2 \psi_2^* \rangle = \langle \psi_1 \psi_1^* \rangle \langle \psi_2 \psi_2^* \rangle + \left| \langle \psi_1 \psi_2^* \rangle \right|^2 + \ldots$$

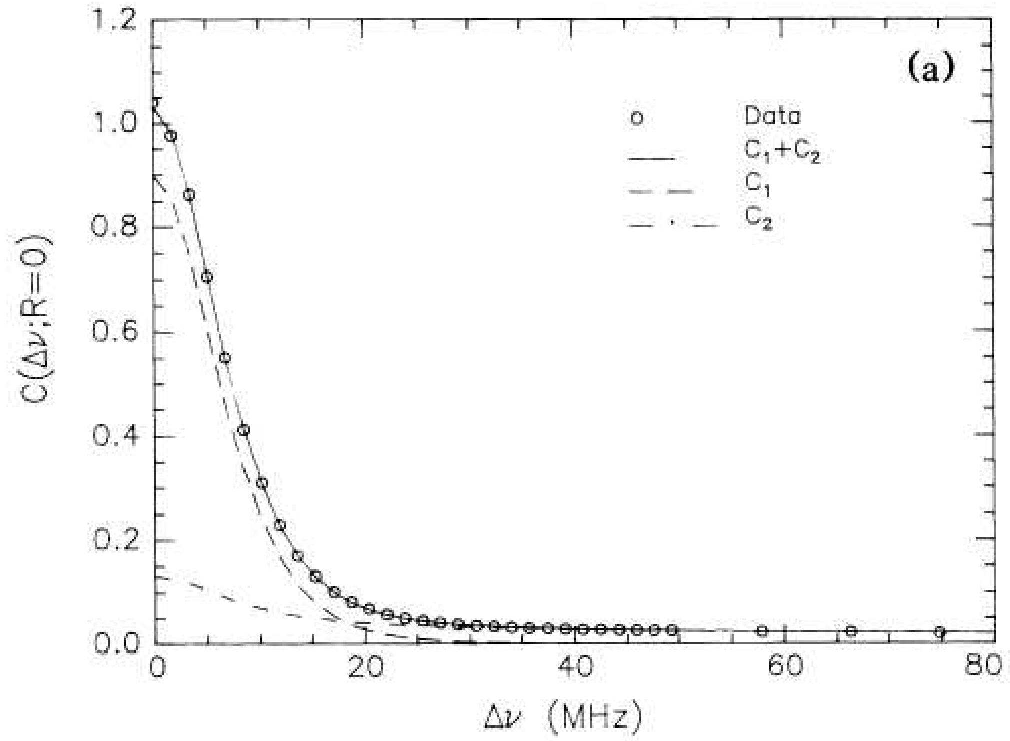
$$\frac{\langle I_1 I_2 \rangle}{\langle I_1 \rangle \langle I_2 \rangle} = 1 + \frac{\left| \langle \psi_1 \psi_2^* \rangle \right|^2}{\langle I_1 \rangle \langle I_2 \rangle} + \cdots = 1 + C_1 + C_2 + C_3$$

Non Gaussian

$$C_1(\Delta\omega) = \left| \frac{\sinh \ell \sqrt{i\Delta\omega L / D + 1/L_a^2} \, \sinh L \sqrt{i\Delta\omega L / D + 1/L_a^2}}{\sinh \ell / L_a \, \sinh L / L_a} \right|^2$$

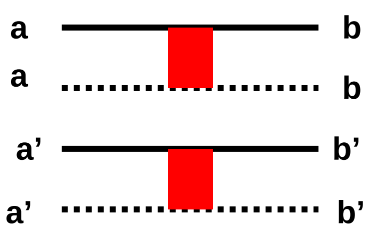
g=10

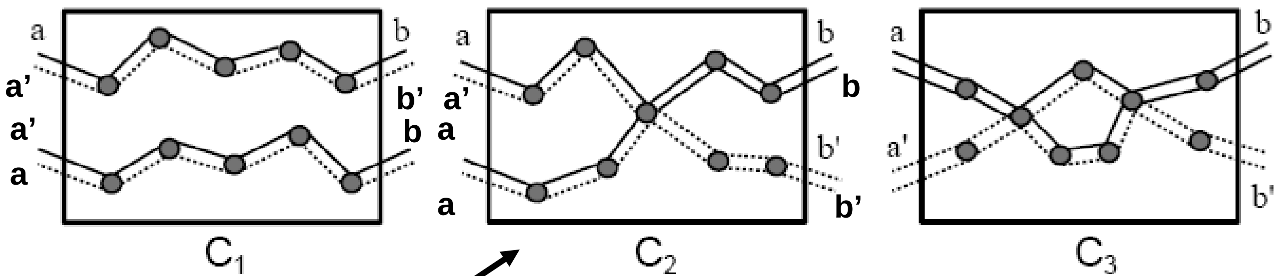
Genack, 1990

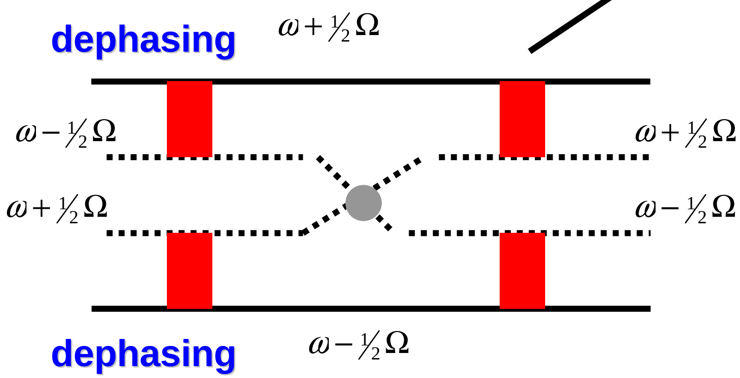


VI.b Long-range and nonGaussian intensity correlations

$$\frac{\langle I_{ab}(-\tfrac{1}{2}\Delta) I_{a'b'}(+\tfrac{1}{2}\Delta)\rangle}{\langle I_{ab}(-\tfrac{1}{2}\Delta)\rangle\langle I_{a'b'}(+\tfrac{1}{2}\Delta)\rangle} = 1\times\delta_{aa}\delta_{bb} + C^1(\Delta)\delta_{aa'}\delta_{bb'} + C^2(\Delta)[\delta_{aa'}+\delta_{bb'}] + C^3(\Delta)$$



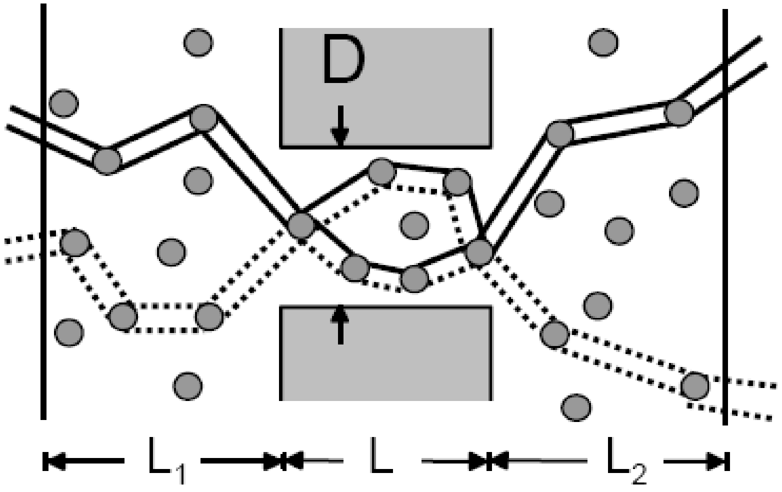




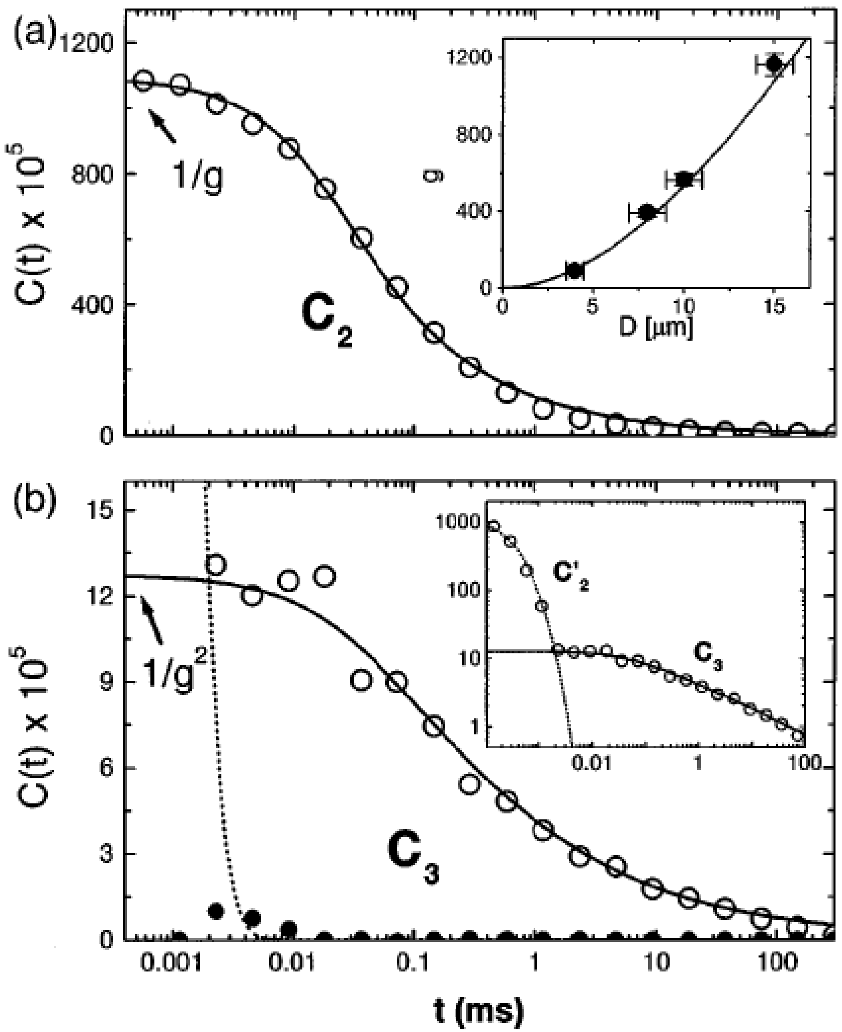
Small long-range

$$C_2(\Omega)\approx\frac{1}{g}\times\frac{1}{L}\int_0^L dz\, C_1(\Omega,z)\approx\frac{1}{g}\times\frac{1}{L}\int_0^L dz\exp-\sqrt{\Omega z^2/D}\propto\frac{1}{g}\frac{1}{\sqrt{\Omega}}$$

Observation of C2 and C3 in dynamic colloids (DWS)



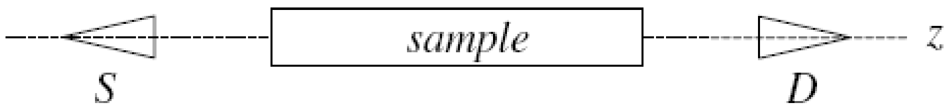
Maret, Scheffold, 1998



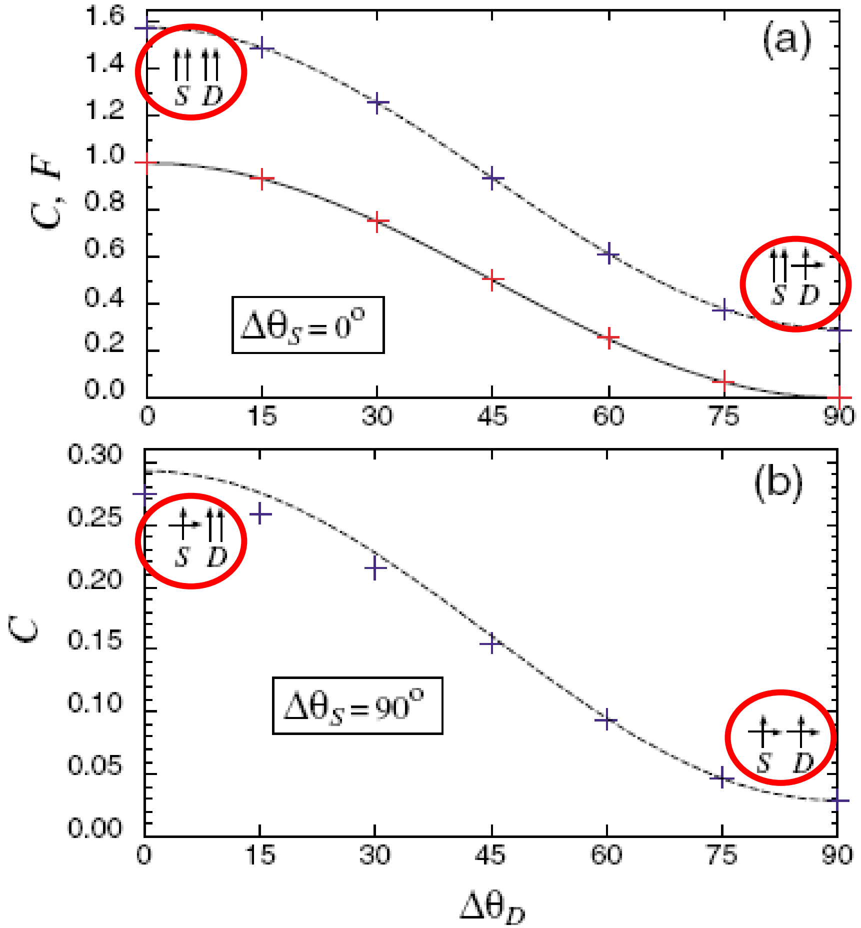
g=100, D= 4 um

Observation of C2 and C3 in microwave polarization

$$\delta_{aa'}^2 \rightarrow \cos^2 \Delta\theta_{S,D}$$



$$C(\Delta\theta_S, \Delta\theta_D) = (1 + A_3)\cos^2\Delta\theta_S \cos^2\Delta\theta_D + A_2\left(\cos^2\Delta\theta_S + \cos^2\Delta\theta_D\right) + A_3$$

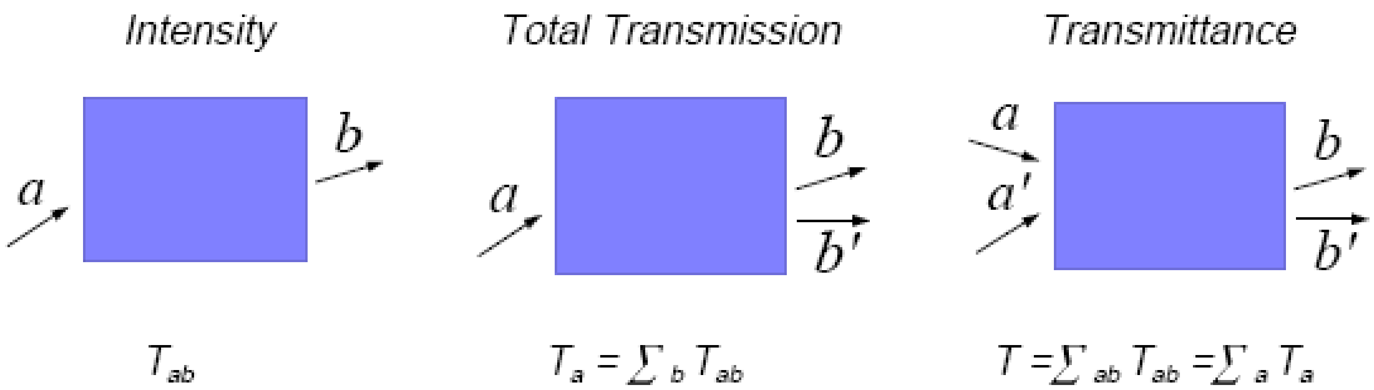


$$A_2' = \frac{2}{3}\frac{1}{g}, \qquad A_3' = \frac{2}{15}\frac{1}{g^2},$$

$$g = 2 \times Ak^2\ell/3\pi L$$

Chabanov etal, 2003)

VI.c Fluctuations of transmission and transmittance



$$\left\langle (\delta T_a)^2 \right\rangle = \left\langle T_{ab} \right\rangle^2 \sum_{bb'=1}^{N} \left(\delta_{bb'} + \frac{1}{g}\left(1+\delta_{bb'}\right) + \frac{1}{g^2} \right)$$

$$= \left(\frac{\ell}{NL}\right)^2 \left(N + \frac{L}{N\ell} N^2 + \left(\frac{L}{N\ell}\right)^2 N^2 \right)$$

$$= \left\langle T_a \right\rangle^2 \left(\frac{1}{N} + \frac{L}{N\ell} + \left(\frac{L}{N\ell}\right)^2 \right)$$

N >>1, g>>1

$$\frac{\left\langle (\delta T_a)^2 \right\rangle}{\left\langle T_a \right\rangle^2} \propto \frac{1}{g}$$

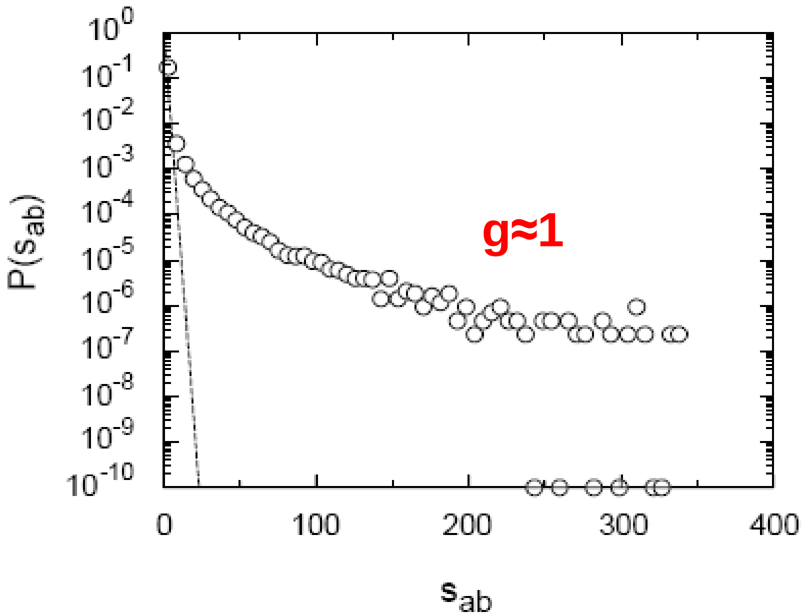
DMPK equation for Q1D

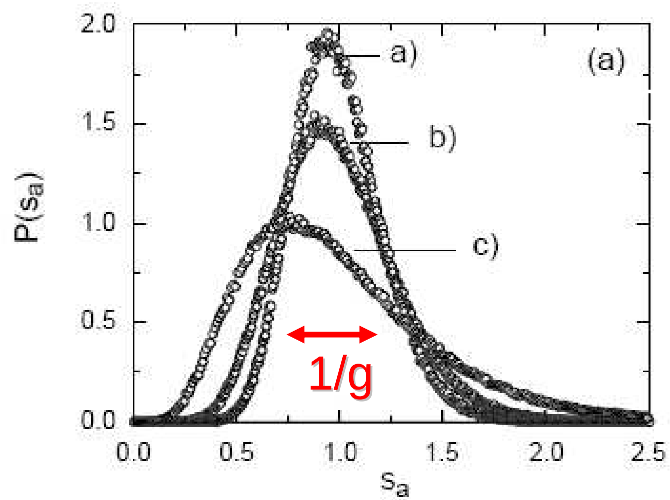
$$P(s_{ab}) = \int_0^\infty \frac{ds_a}{s_a} P(s_a) \exp(-s_{ab}/s_a).$$

$$P(s_a) = \int_{-i\infty}^{i\infty} \frac{dx}{2\pi i} \exp\left(x s_a - \Phi(x)\right),$$

«in a fixed speckle patteren (T_a fixed), the field Is Gaussian distributed »

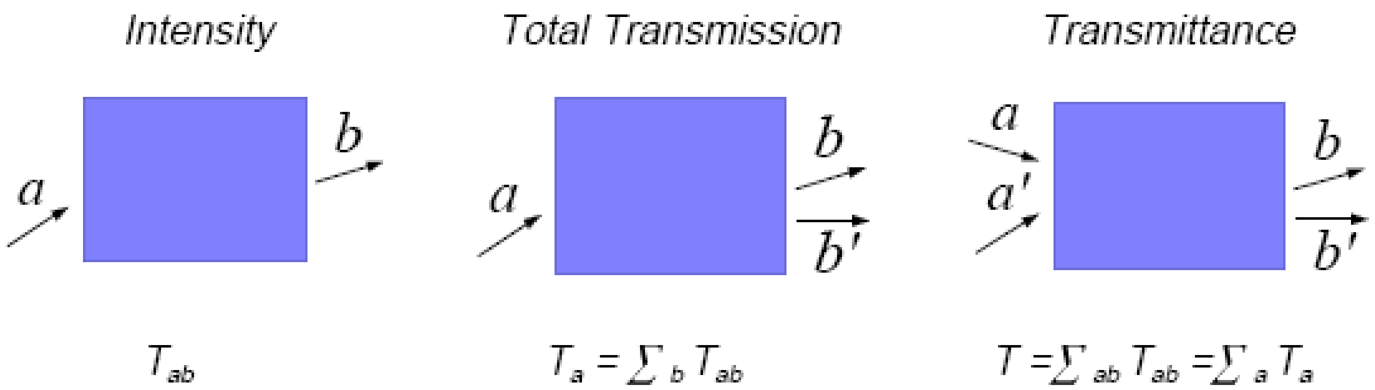
$$\Phi(x) = g \ln^2(\sqrt{1 + x/g} + \sqrt{x/g})$$





GHz frequencies, Genack, 2000

Fluctuations of transmission and transmittance



$$\left\langle (\delta T)^2 \right\rangle = \left\langle T_{ab} \right\rangle^2 \sum_{aa'bb'=1}^{N} \left(\delta_{aa'}\delta_{bb'} + \frac{1}{g}\left(\delta_{aa'} + \delta_{bb'}\right) + \frac{1}{g^2} \right)$$

$$= \left(\frac{\ell}{NL} \right)^2 \left(N^2 + \frac{L}{N\ell} N^3 + \left(\frac{L}{N\ell} \right)^2 N^4 \right)$$

$$= \left(\left(\frac{\ell}{L} \right)^2 + \frac{\ell}{L} + 1 \right)$$

N >>1, g>>1

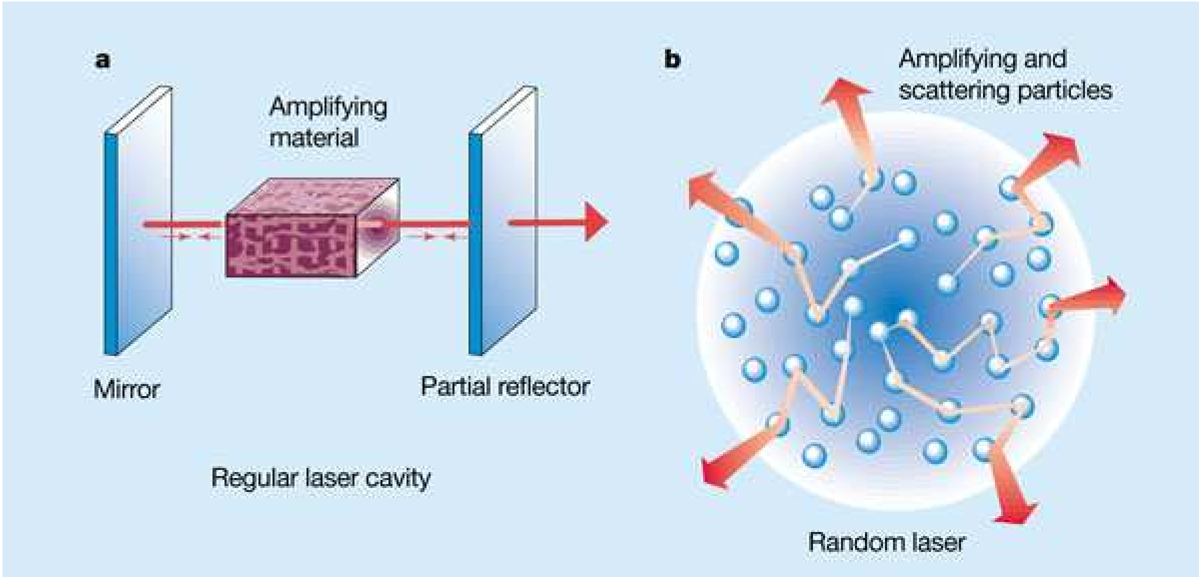
$$\left\langle \delta T^2 \right\rangle \propto 1$$

Universal conductance fluctuations (never observed with light)

VI. Speckles and correlations

conclusion: near g=1, fluctuations in transmission are
are strong and non Gaussian

VII. Random Laser



Generation of spatially incoherent short pulses in laser-pumped neodymium stoichiometric crystals and powders

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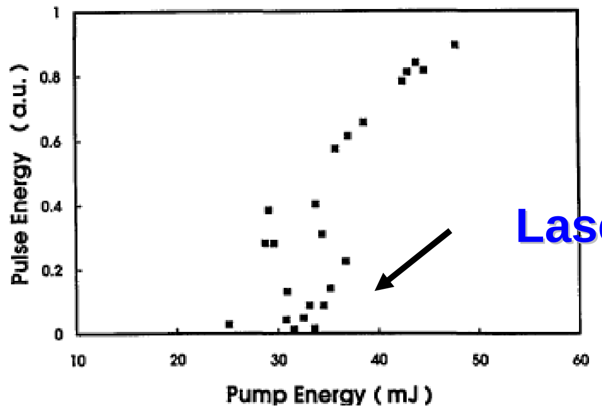
Received August 10, 1992; revised manuscript received July 13, 1993

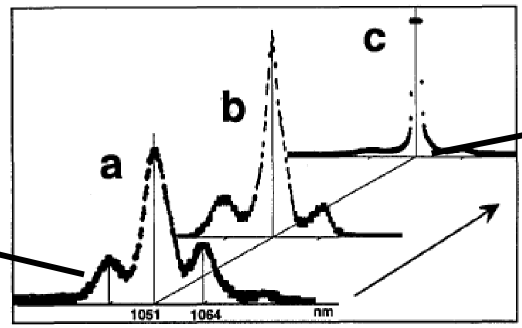
$NdCl_3 \cdot 6H_2O$

Hydrated Neodynium Chloride powder



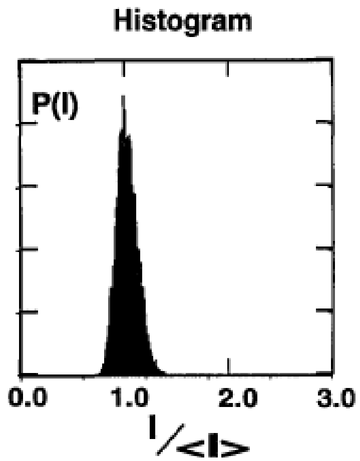
Laser threshold ?

Amplified Stimulated emission



Narrow laser spikes ? (1.15 nm)

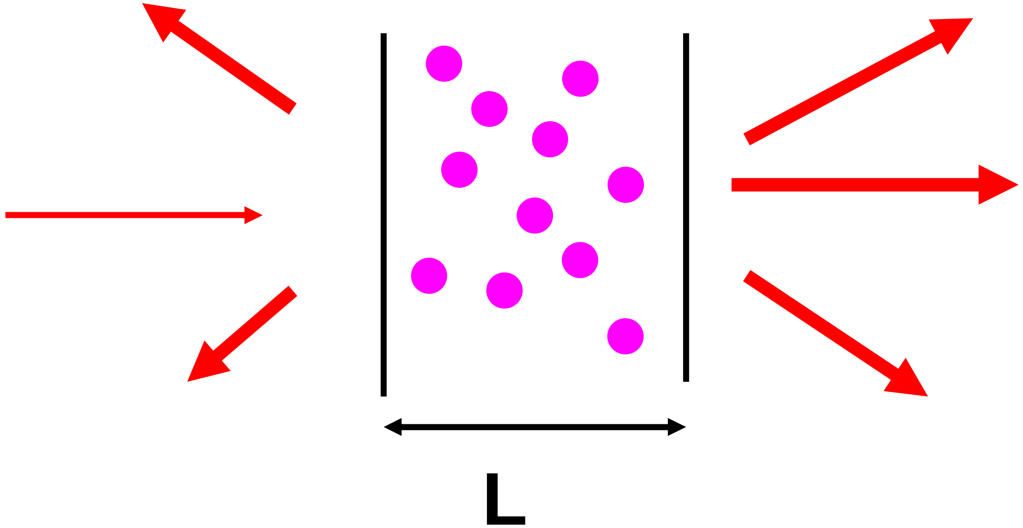
$NdCl_3;6H_20$	$\frac{\sigma}{<I>}$: 14%
NdLaPP	$\frac{\sigma}{<I>}$: 8%



$$\frac{\Delta I}{\langle I \rangle} = \frac{1}{\sqrt{N}} \Rightarrow N \approx 50$$

$$\tau_c = \frac{\lambda^2}{c_0 \Delta\lambda} \approx 6\,\text{ps} \Rightarrow N = \frac{T}{\tau_c} = \frac{300\text{ps}}{6\text{ps}} = 50$$

$$\partial_t \rho(\mathbf{r},t) - D\nabla^2\rho(\mathbf{r},t) - \frac{\rho(\mathbf{r},t)}{\tau_{\text{gain}}} = S\,\delta(t)\,\delta(\mathbf{r}-\mathbf{r}_S)$$



Leak rate > Level spacing

$$\frac{D}{L^2} > \frac{1}{L^3\rho(\omega)}$$

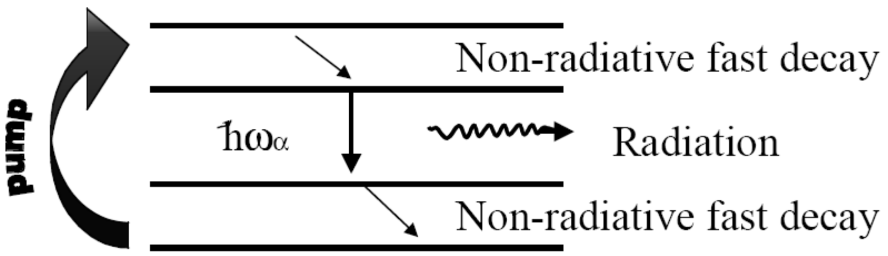
→ Leaky modes overlap

« Incoherent »
Laser action
« photonic bomb ») if

$$\frac{D}{L^2} < \frac{1}{\tau_{\text{gain}}}$$

Threshold depends on mean free path

Letokhov, 1968



Pump light

$$\frac{\partial W_G(\vec{r},t)}{\partial t} = D\nabla^2 W_G(\vec{r},t) - \sigma_{abs} v[N_t - N_1(\vec{r},t)]W_G(\vec{r},t) + \frac{1}{l_G} I_G(\vec{r},t),$$

Probe light

$$\frac{\partial W_R(\vec{r},t)}{\partial t} = D\nabla^2 W_R(\vec{r},t) + \sigma_{em} v N_1(\vec{r},t) W_R(\vec{r},t) + \frac{1}{l_R} I_R(\vec{r},t),$$

Amplified stimulated emission

$$\frac{\partial W_A(\vec{r},t)}{\partial t} = D\nabla^2 W_A(\vec{r},t) + \sigma_{em} v N_1(\vec{r},t) W_A(\vec{r},t) + \frac{1}{\tau_e} N_1(\vec{r},t),$$

Lasing level population

$$\frac{\partial N_1(\vec{r},t)}{\partial t} = \sigma_{abs} v[N_t - N_1(\vec{r},t)]W_G(\vec{r},t) - \sigma_{em} v N_1(\vec{r},t)[W_R(\vec{r},t) + W_A(\vec{r},t)] - \frac{1}{\tau_e} N_1(\vec{r},t).$$

Basic questions

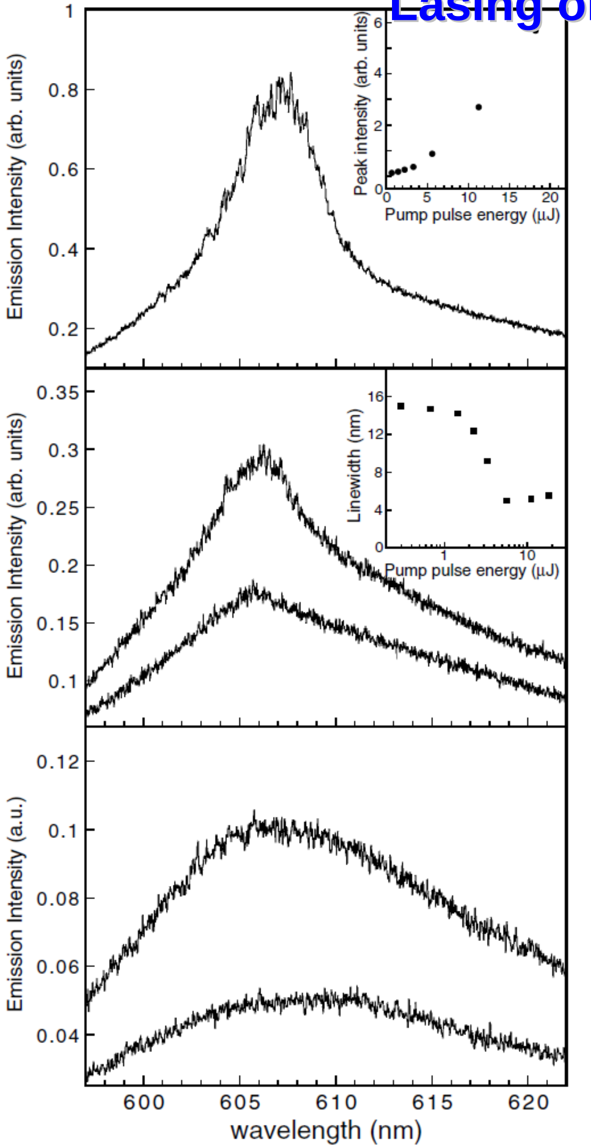
- Is this « random laser » really a laser
  (or just stimulated emission amplified by scattering ASE)?
- What happens beyond diffuse threshold?
- Is it possible to use one single mode to lase?
- How does the random cavity look like? Localized, extended?

What is a laser anyway, and what makes a laser « random »?

rhodamine 640 dye solutions with ZnO nanoparticles

n = 3 10^{11} /cm^3 **Incoherent Lasing or ASE?**

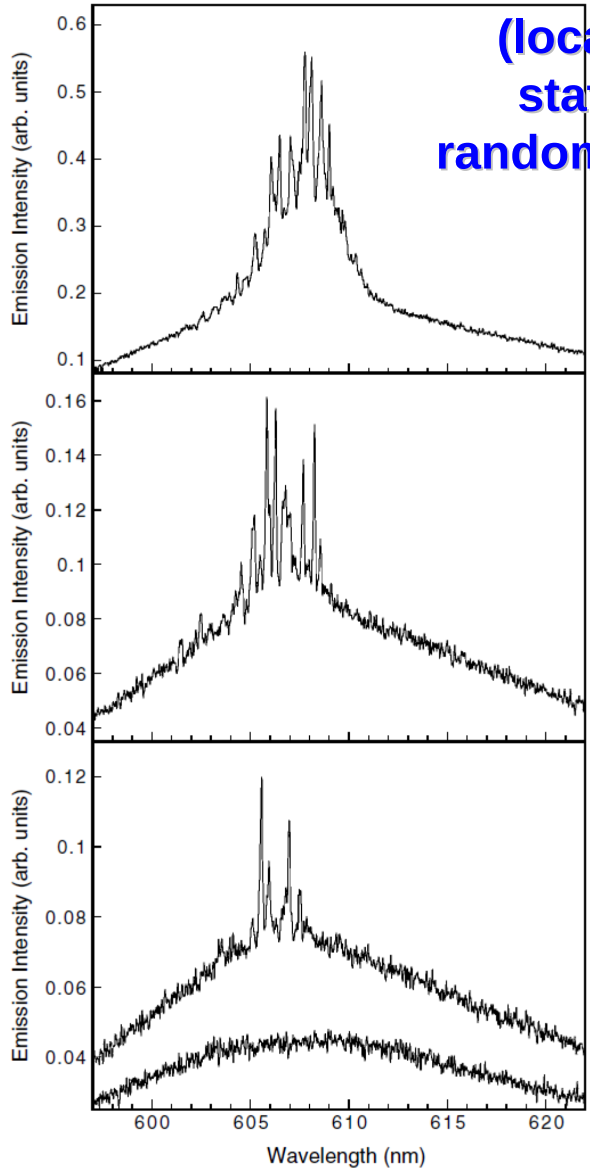
$k\ell \approx 5$



Increasing pump

n = 1 10^{12} /cm^3 **Coherent lasing ? (localized state as random cavity)**

$k\ell \approx 1.5$

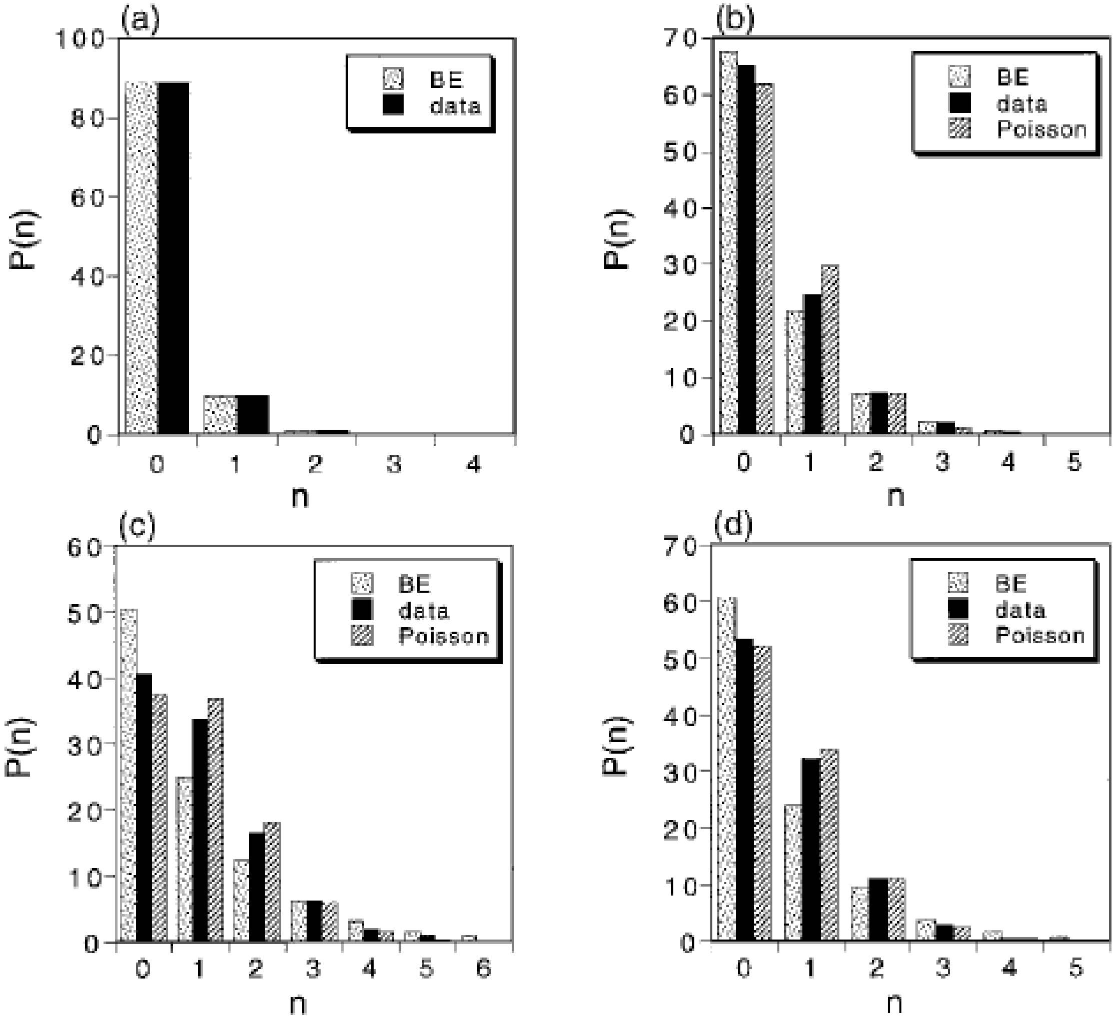


Cao, 1990

Photon statistics

I_{pump}=1.0 $I_{threshold}$

I_{pump}=1.5 $I_{threshold}$

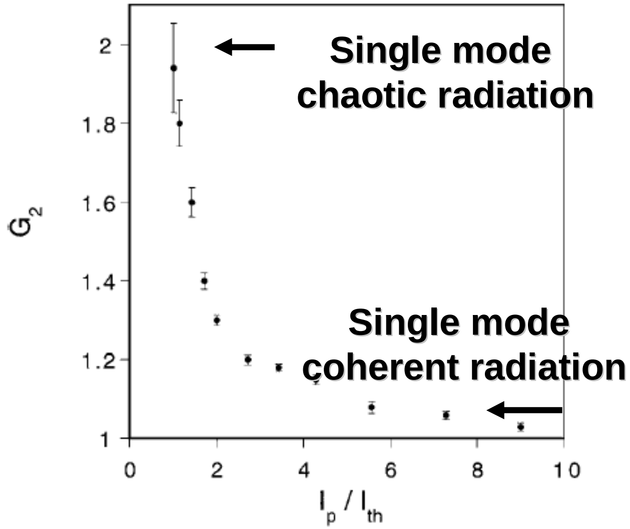


I_{pump}=3.0 $I_{threshold}$

I_{pump}=5.6 $I_{threshold}$

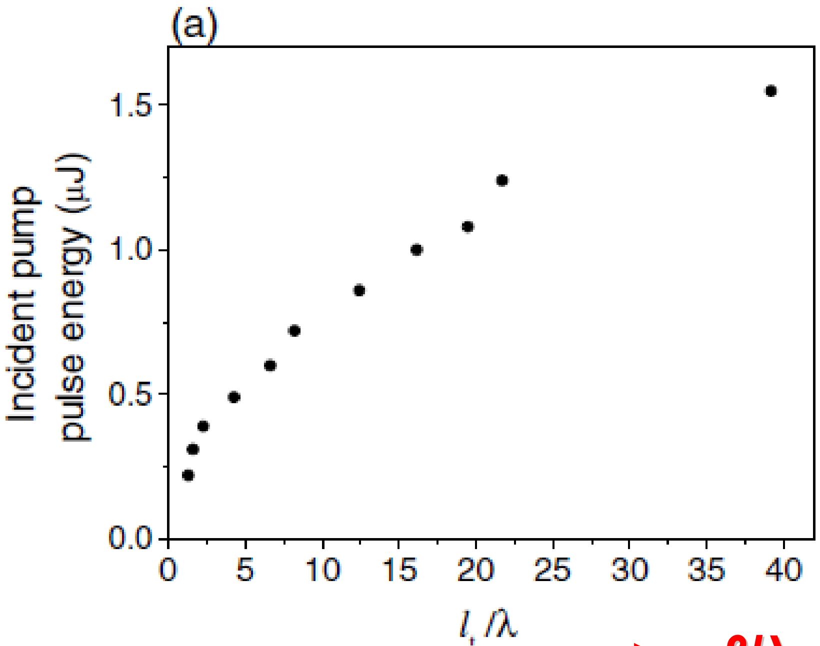
kℓ≈1.5

$$G_2 = 1 + \frac{\langle(\Delta n)^2\rangle - \langle n\rangle}{\langle n\rangle^2}.$$



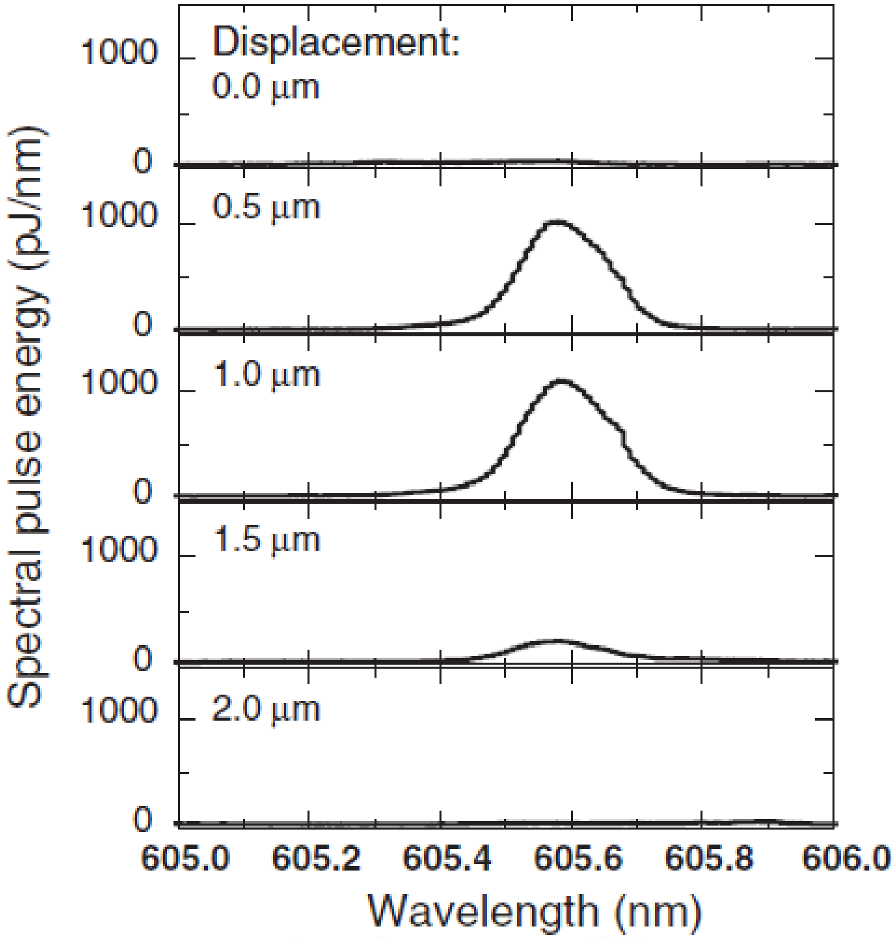
Cao, 1990

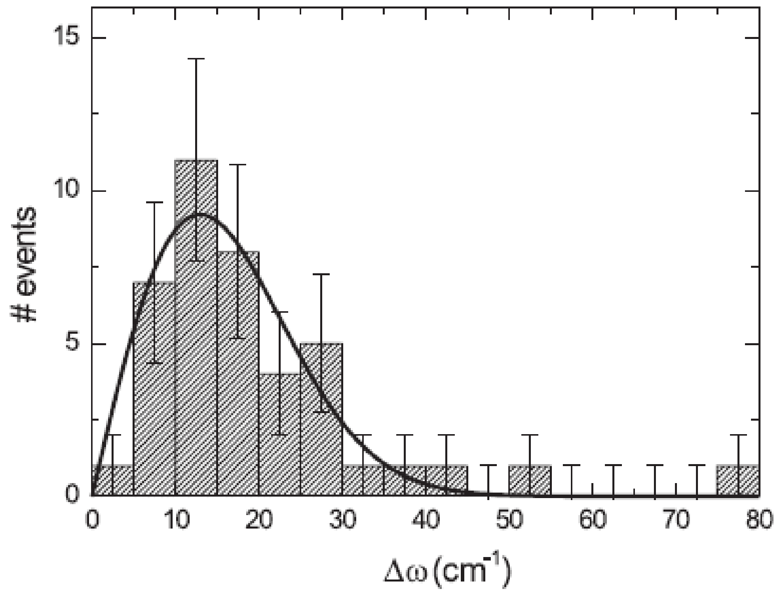
threshold



ℓ/λ

Nature of lasing modes





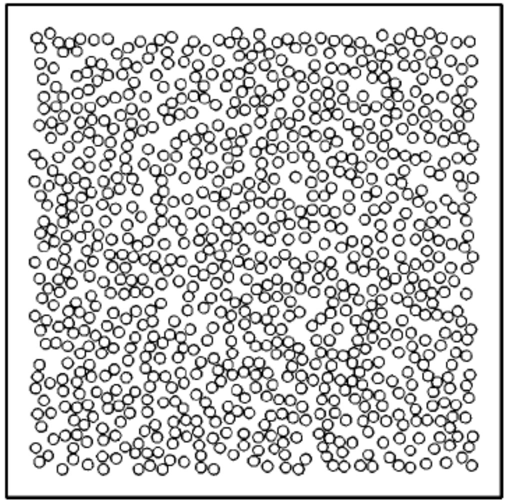
Level repusion between lasering modes. Not localized after all?

Pumping beam at different spot: Spatial extent of 2 µm: local modes not extended?

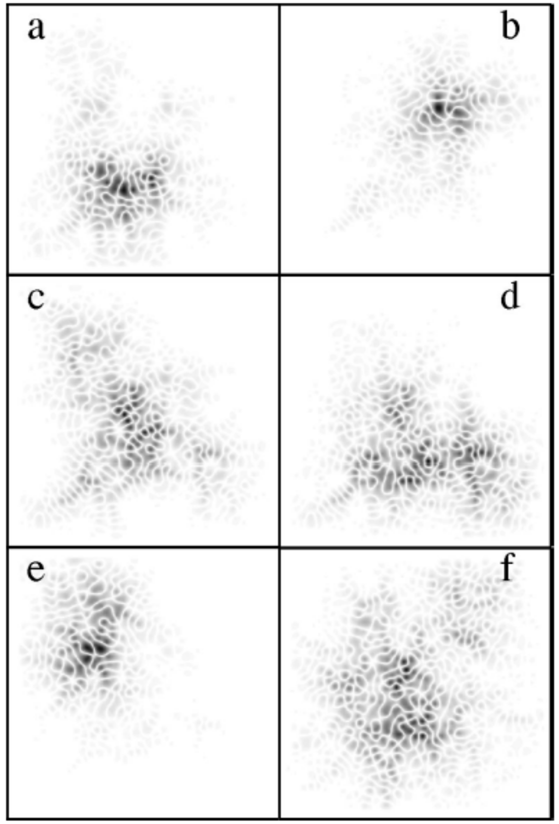
$k\ell$=6.4

Van der Molen, Lagendijk, 2007

Nature of lasing modes



No gain



pump

Localized states act as inert lasing cavities

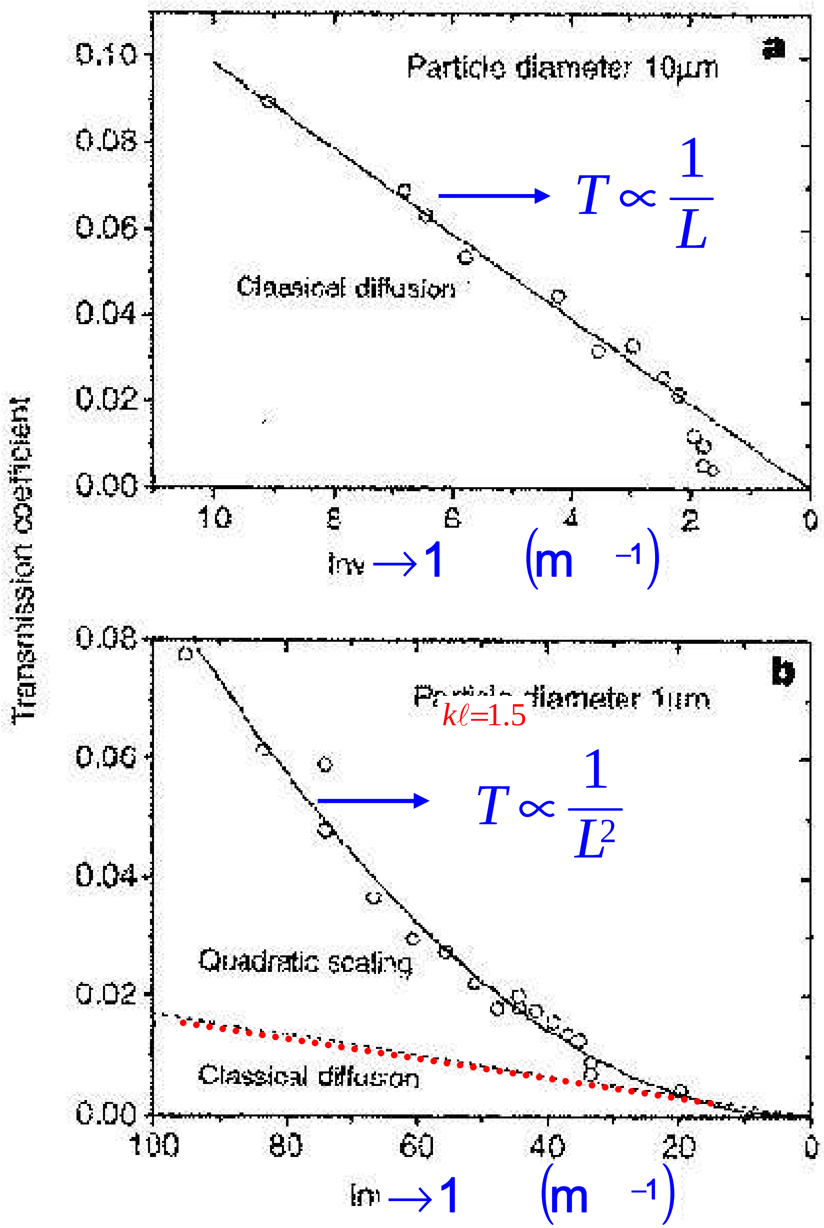
Sebbah, Vanneste, 2002

VIII. Observation & Modelling of Anderson Localization in high dimensions

VIII.a 3D Localization of Light

Photonic conductor

Photonic insulator? or Photonic Absorber?



$k\ell=76$

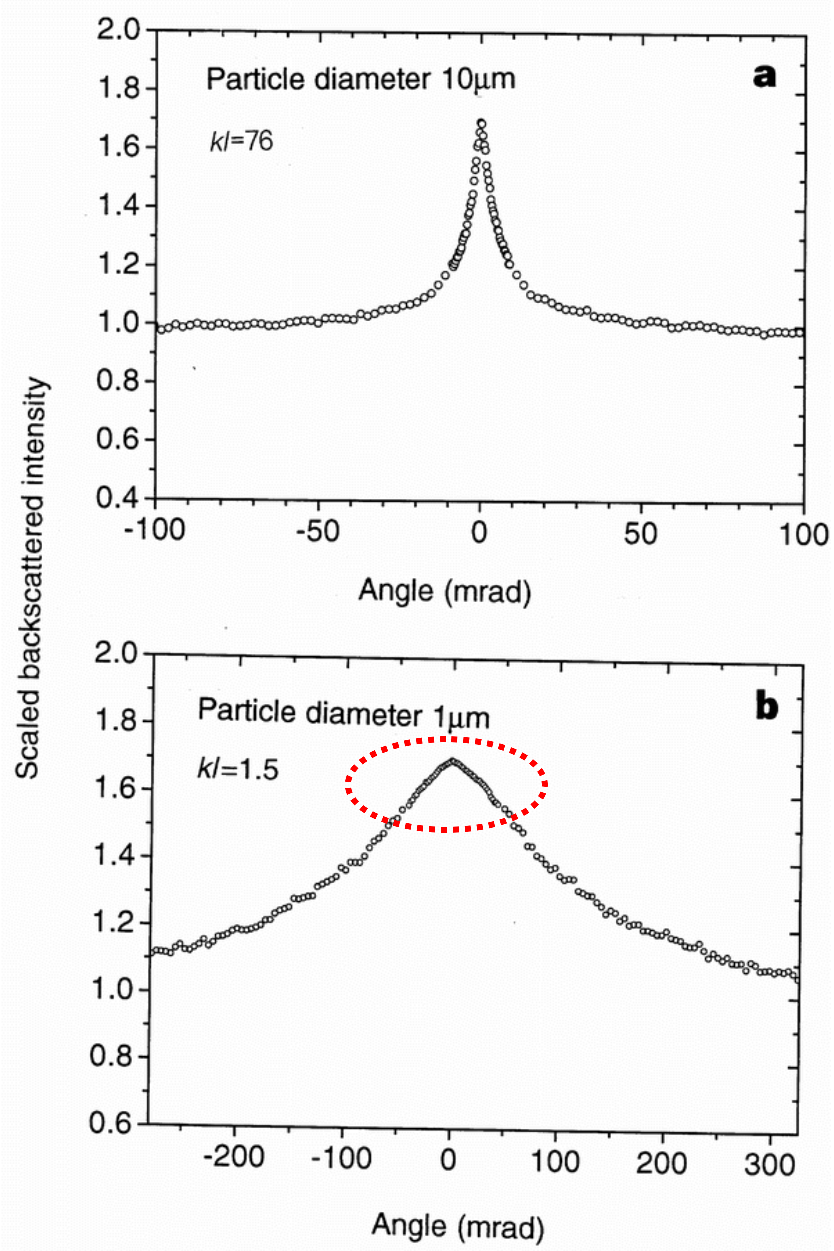
$k\ell=1.5$

GaAs powder

D=ℓ²/L ?

Wiersma, Lagendijk, Nature 1997

VIII.a 3D Localization of Light



$k\ell=76$

$k\ell=1.5$

Enhanced backscattering in localized regime?

Wiersma, Lagendijk, Nature 1997

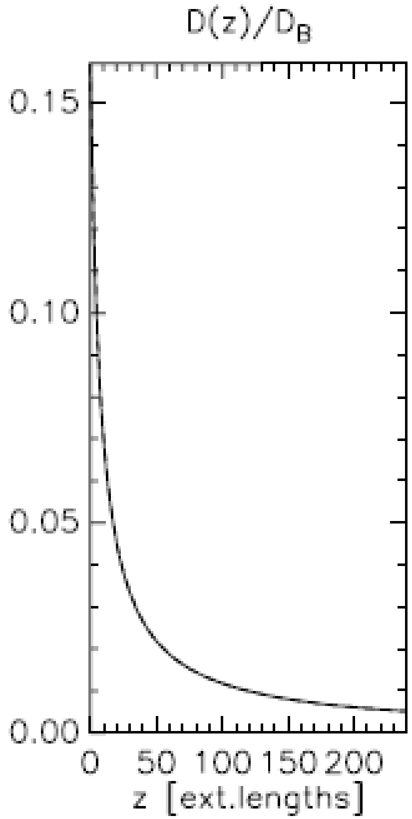
$$D(z)^{-1} = D_B^{-1} + \frac{2}{k^2\ell}\int_0^{1/\ell} dq_\parallel q_\parallel C(z,z,q_\parallel), \quad (3a)$$

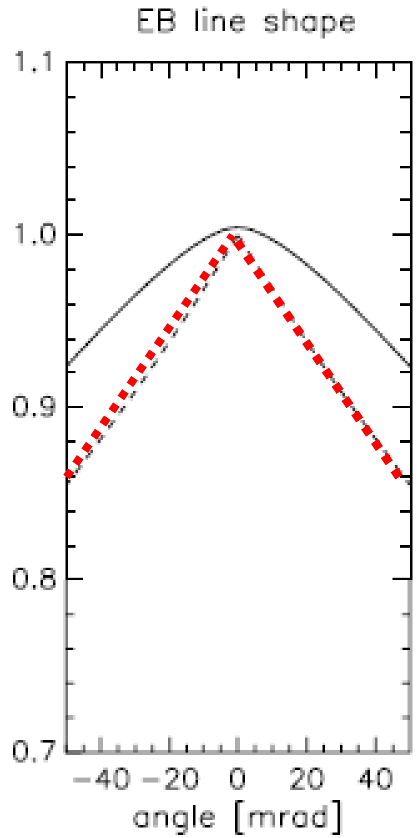
$$-\partial_z D(z)\partial_z C(z,z',q_\parallel) + D(z)q_\parallel^2 C(z,z',q_\parallel) = \frac{4\pi}{\ell}\delta(z-z'), \quad (3b)$$

$$C(0,z',q_\parallel) - z_e(0)\partial_z C(0,z',q_\parallel) = 0, \quad (3c)$$

$$C(L,z',q_\parallel) + z_e(L)\partial_z C(L,z',q_\parallel) = 0. \quad (3d)$$

$k\ell=1$





Akkermans, Maynard

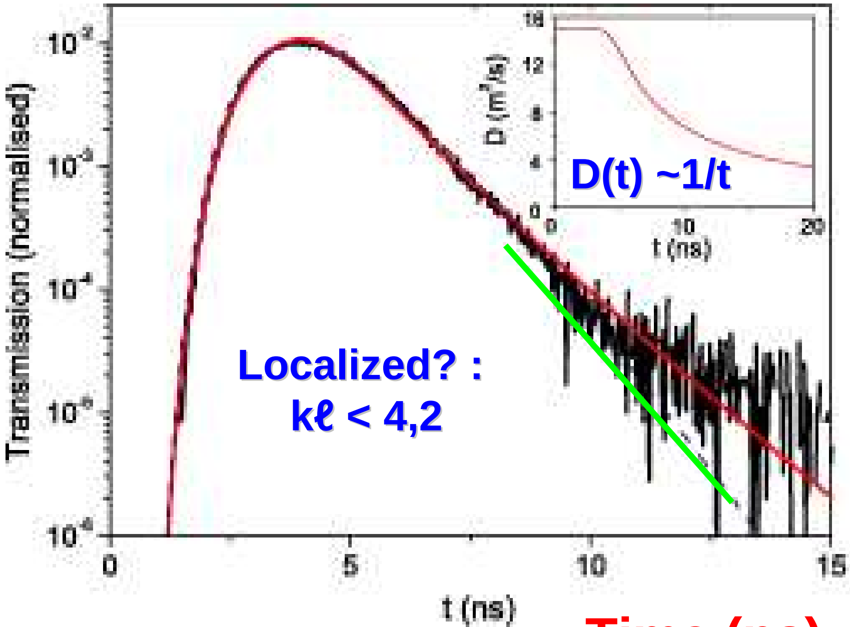
$$D(z) = \frac{D(0)}{1+z/\zeta} \Rightarrow T \propto \frac{1}{L^2}$$

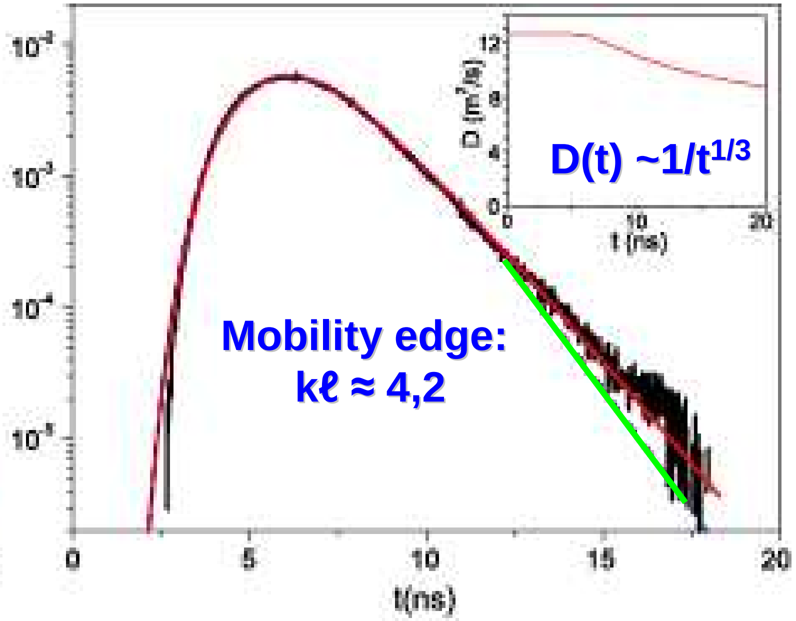
$$I_c(\theta) \approx \frac{1}{1 - z_e(0)/\xi + z_e(0)\sqrt{q_\parallel^2 + 1/\xi^2}},$$

Van Tiggelen etal, 2000

VIII.a 3D Localization of Light

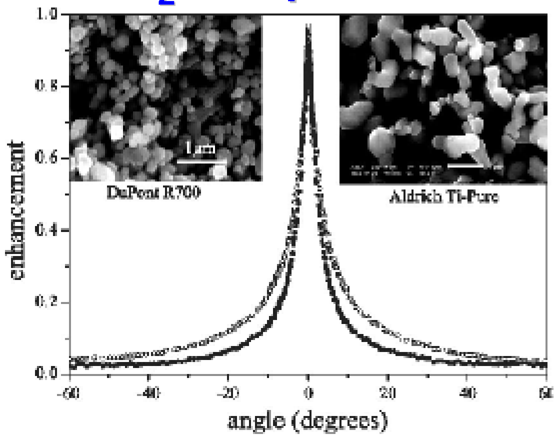
transmission





Time (ns)

TIO$_2$ sample and CBS

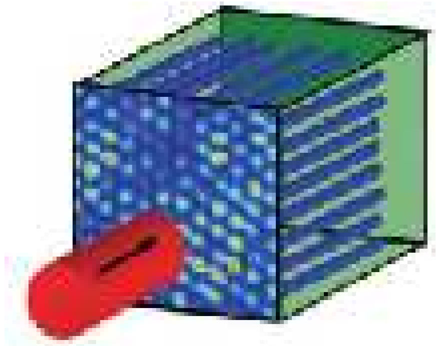


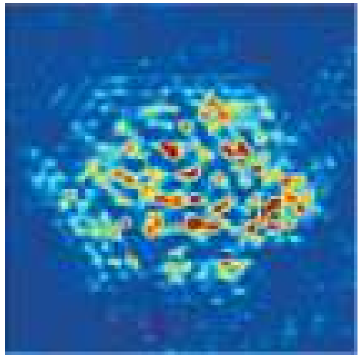
$$\partial_t I(\mathbf{r},t) - D(t)\Delta I(\mathbf{r},t) = S\delta(t)\delta(\mathbf{r})$$

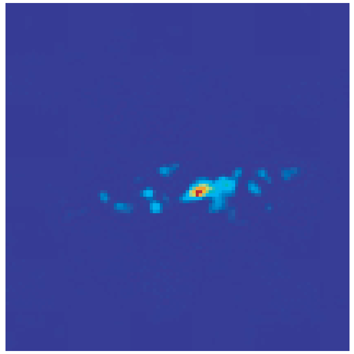
$$T(t) \propto \exp\left[-\frac{1}{L^2}\int_0^t dt D(t)\right]$$

Maret etal, EPL 2006

VIII.b 2D Transverse localisation of light

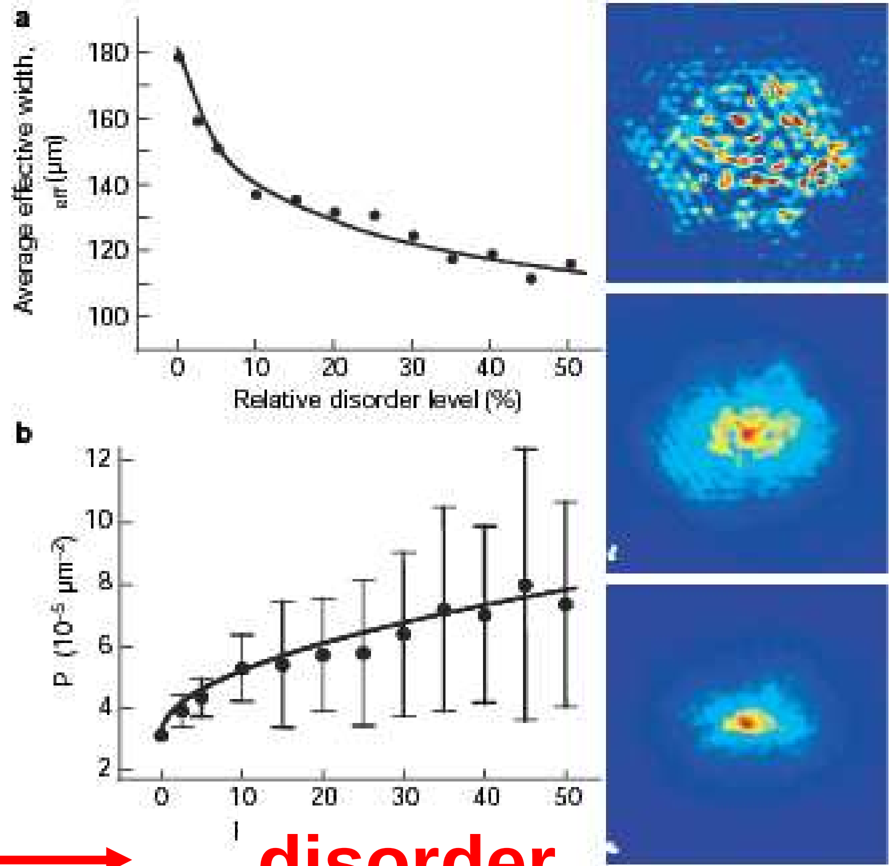






Width (μm)

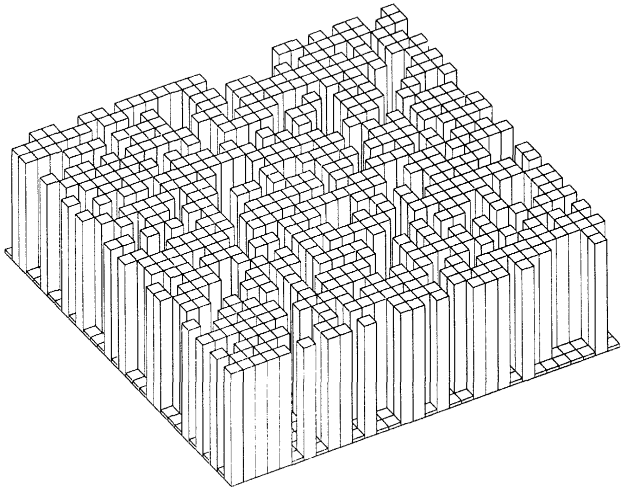
Inverse participation ratio



disorder

Schwartz, Segev, etal
Nature 2007

VIII.b 2D Transverse localisation of light



$$\varepsilon(\boldsymbol{\rho})\frac{\omega^2}{c_0^2}\Psi(z,\boldsymbol{\rho})+\Delta\Psi(z,\boldsymbol{\rho})=0$$

$$\Psi(z,\boldsymbol{\rho})=\exp(ikz)\Phi(z,\boldsymbol{\rho})$$

$$\left(\frac{\omega^2}{c_0^2}-k^2+\delta\varepsilon(\boldsymbol{\rho})\frac{\omega^2}{c_0^2}\right)\Phi(z,\boldsymbol{\rho})+\Delta_\rho\Phi(z,\boldsymbol{\rho})+\partial_z^2\Phi(z,\boldsymbol{\rho})=\frac{2k}{i}\partial_z\Phi(z,\boldsymbol{\rho})$$

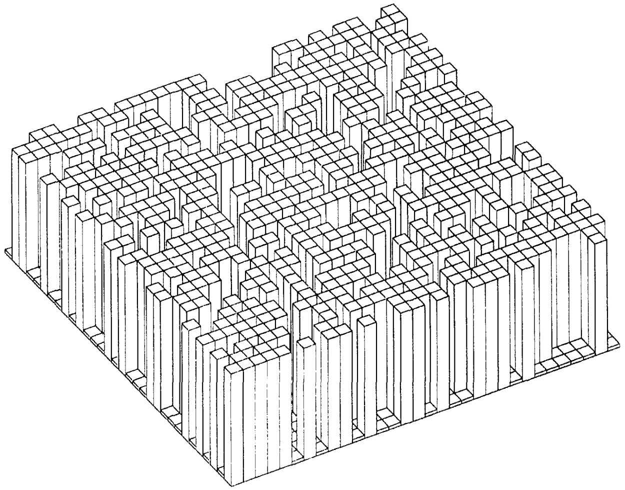
$$\left|\partial_z^2\Phi\right| << \left|k\partial_z\Phi\right|$$

Paraxial approximation

2D Schrödinger equation with « time » z :
localization strongest at small « energies »
→ Wave packet transversly localized

De Raedt, Lagendijk and De Vries 1988,

VIII.b 2D Transverse localisation of light



$$\left(\frac{\omega^2}{c_0^2} - k^2 + \delta\varepsilon(\boldsymbol{\rho})\frac{\omega^2}{c_0^2}\right)\Phi(z,\boldsymbol{\rho}) + \Delta_\rho \Phi(z,\boldsymbol{\rho}) = \frac{2k}{i}\partial_z \Phi(z,\boldsymbol{\rho})$$

$$\overline{k}_\rho^2 \approx \left(\frac{\omega}{c_0}\right)^2 - k^2 << \left(\frac{\omega}{c_0}\right)^2$$

$$\frac{\overline{k_\rho}}{\ell} = n\pi\int \frac{d^2\mathbf{k}_\rho}{(2\pi)^2}|V_{\mathbf{kk'}}|^2 \delta\left(\frac{\omega^2}{c_0^2} - k^2 - k_\rho{}^2\right)$$

$$= \frac{1}{2}n(\delta\varepsilon)^2 A^2 \frac{\omega^4}{c_0^4}$$

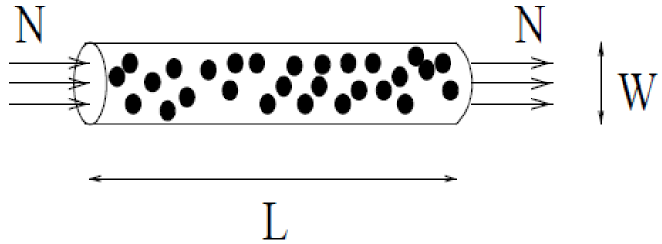
2D localization length

$$\xi = \ell \exp\left(\frac{\pi}{2}\overline{k}_\rho \ell\right)$$

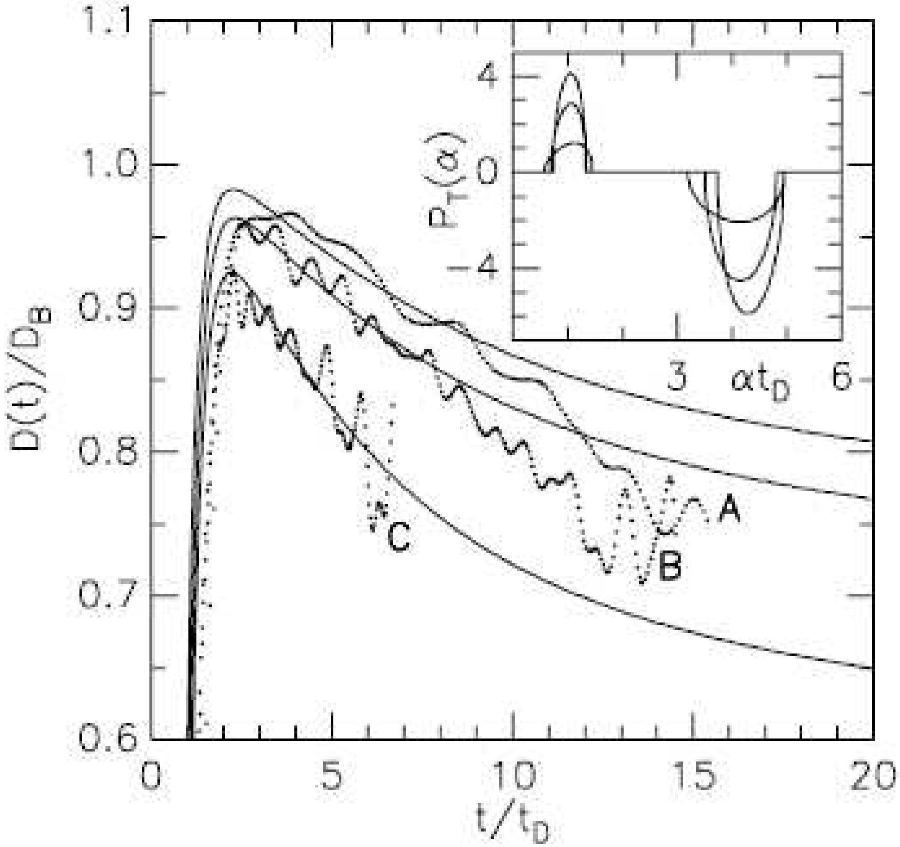
$$\Rightarrow \overline{k}_\rho \ell = \frac{2}{f\delta\varepsilon^2}\frac{1}{Ak^2}\left(\frac{\omega^2}{c_0^2} - k^2\right)$$

$\delta\varepsilon \approx 10^{-4}$ **!** **but** $\omega / c_0 \approx k$

VIII.c Quasi 1D localisation of microwaves



D(t)=-d/dt logT(t)



$$I_{T,R}(t) = \int_0^\infty d\alpha \exp(-\alpha t) P_{T,R}(\alpha).$$

Leakage rate distribution

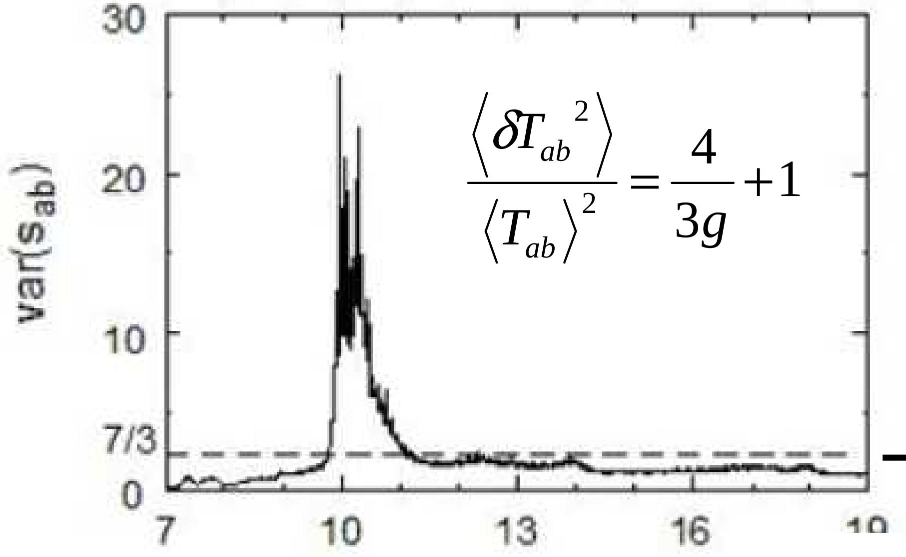
$$g = 2 \times Ak^2\ell/3\pi L$$

=9,7,4

Data: A.Z. Genack (2004); theory: Skipetrov and van Tiggelen

VIII.c Quasi 1D localisation of microwaves

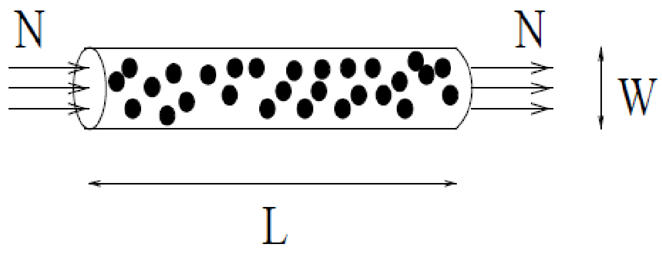
Fluctuations in
channel transmission



$$\frac{\langle \delta T_{ab}{}^2 \rangle}{\langle T_{ab} \rangle^2} = \frac{4}{3g} + 1$$

→ g=1

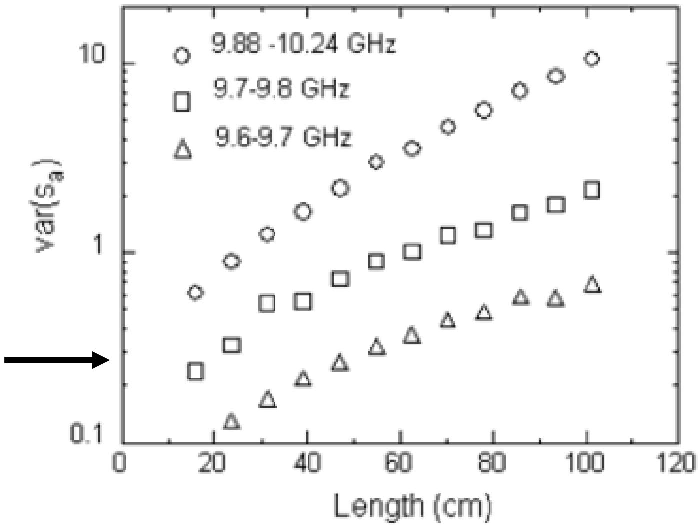
→ f (GHz)



$$g = 2 \times Ak^2\ell/3\pi L$$

Fluctuation in
total transmission

g=1 →



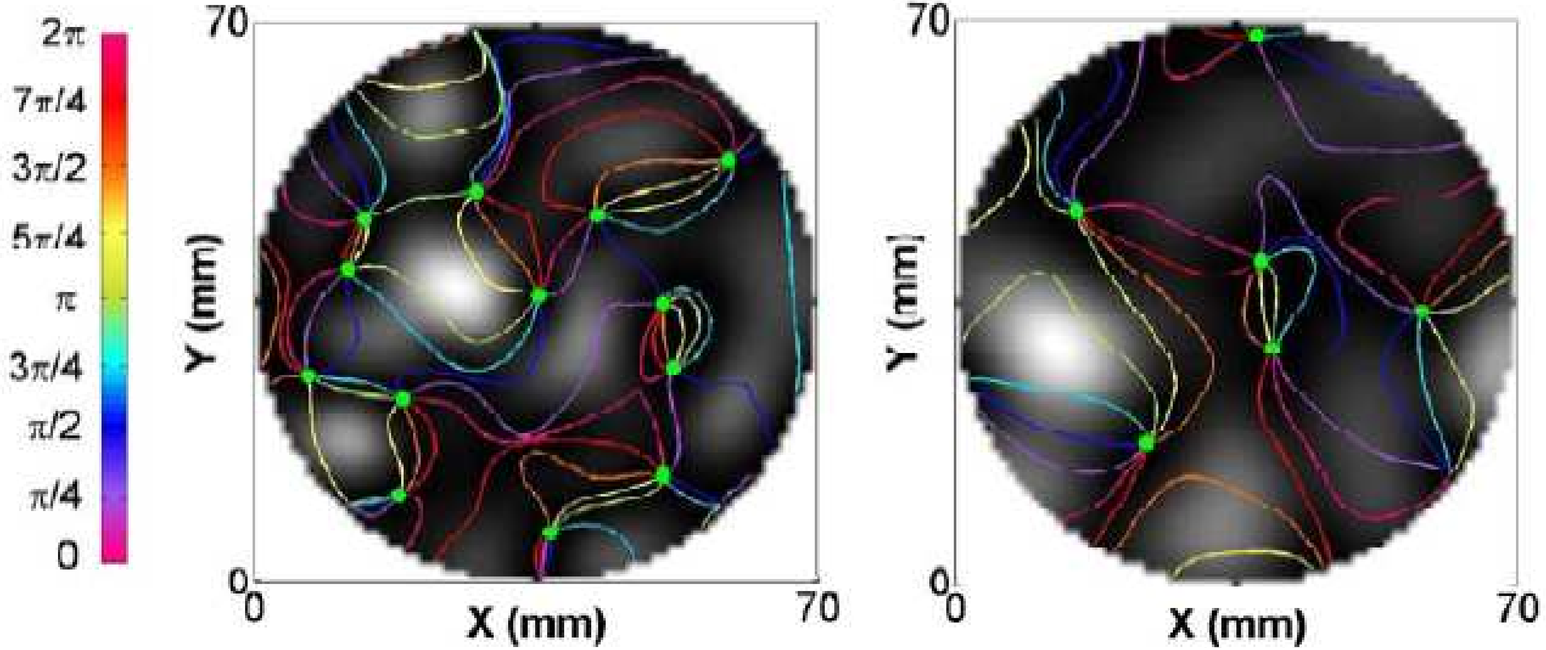
→ L

A.Z. Genack (2004)

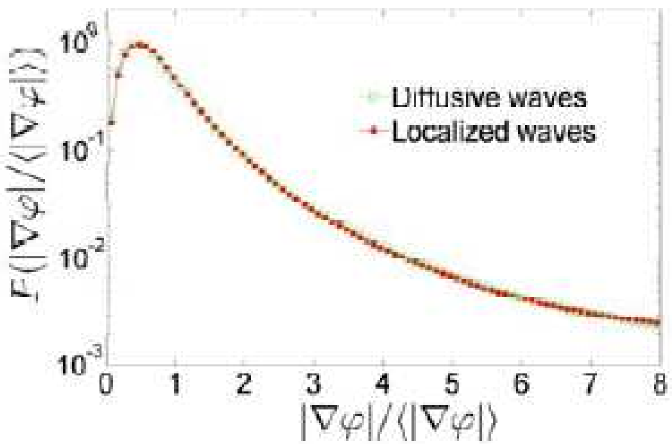
VIII.c Quasi 1D localisation of microwaves

phase

diffuse **localized**



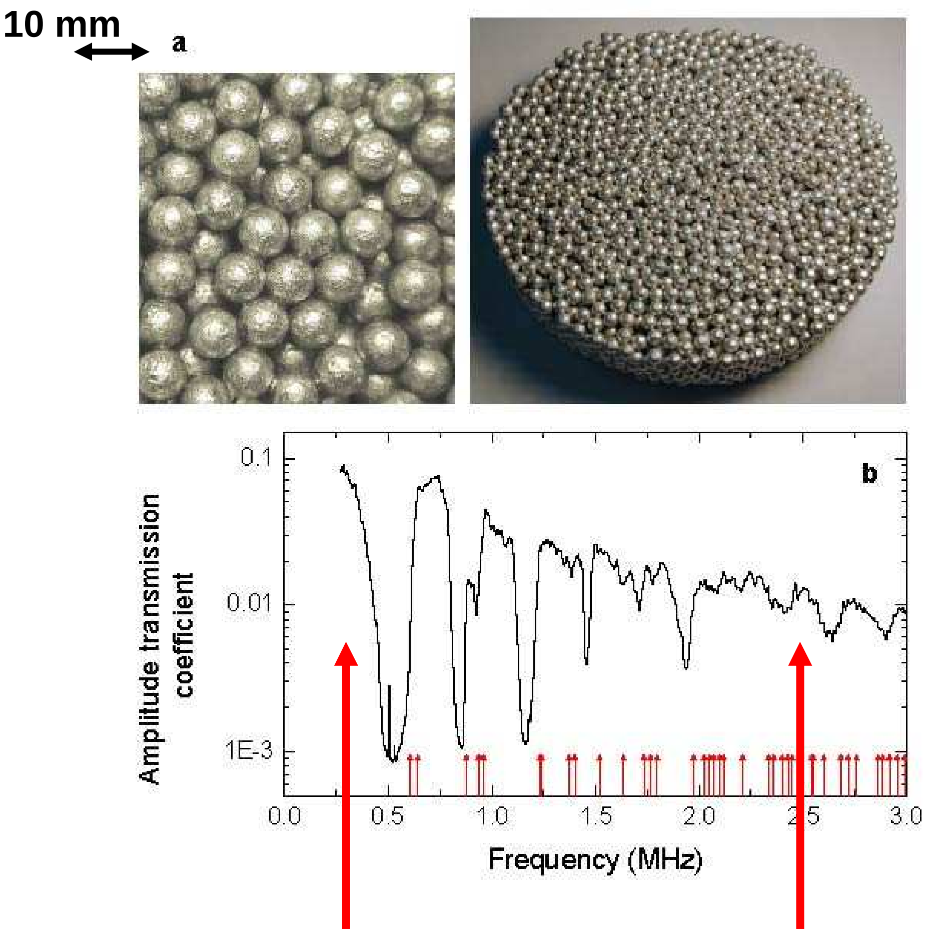
$$P\left(\left|\frac{d\phi}{dr}\right|\right)$$



Data: A.Z. Genack (2004);
Theory: Van Tiggelen

bvt3

VIII.d 3D localisation of ultrasound



8 – 23 mm

Diffuse λ_p=9 mm, ℓ=2 mm

$k_p\ell$=1.4

Localized λ_p=2 mm, ℓ=0.6 mm

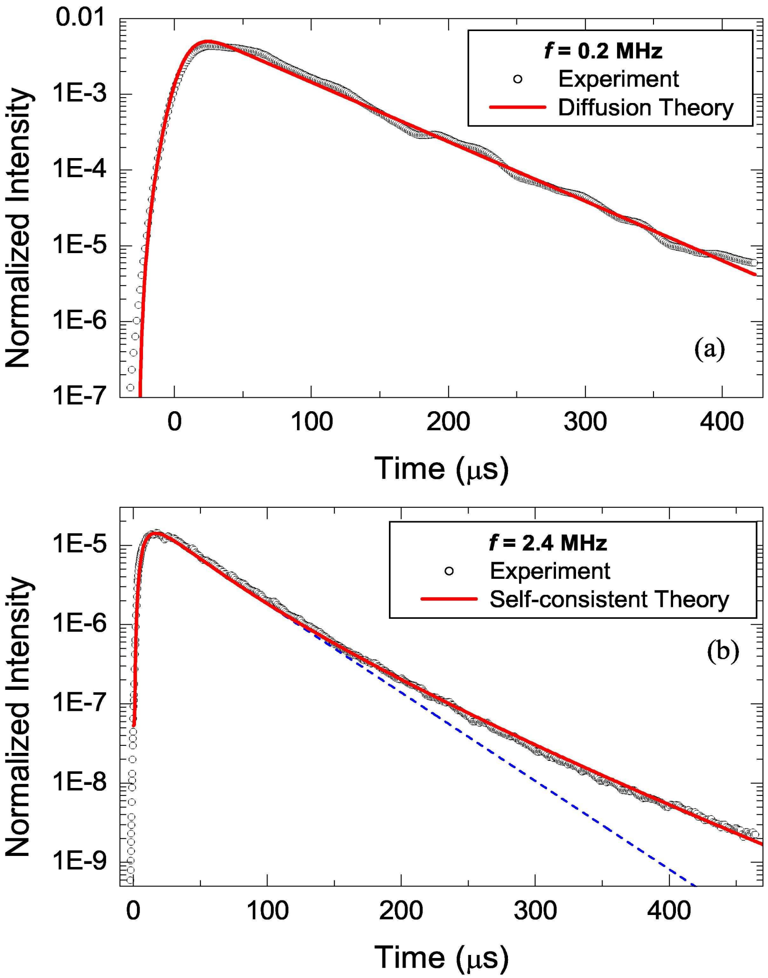
$k_p\ell$=1.9

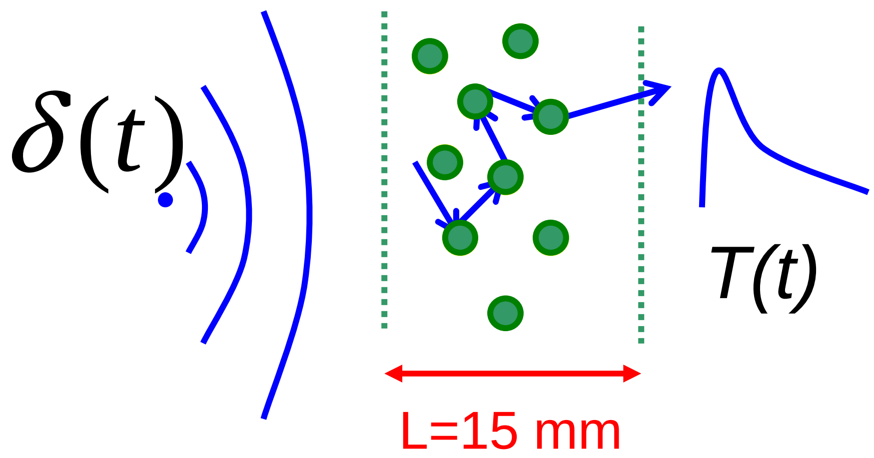
Page , Skipetrov, Van Tiggelen, Nature Physics 2008

Slide 51

bvt3 bart van tiggelen; 09/01/2008

Time-dependent transmission of ultrasound



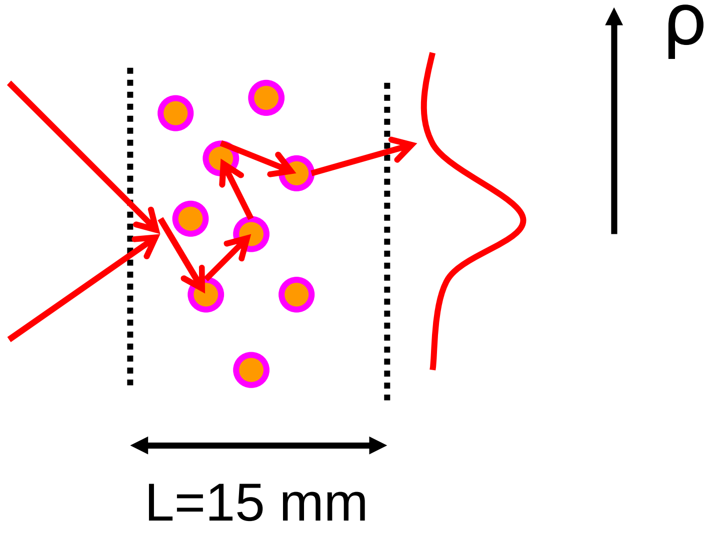


D(t) ?

$$-\partial_t \rho(\mathbf{r},t) - D(t)\nabla^2 \rho(\mathbf{r},t) = S(\mathbf{r})\delta(t)$$

bvt4

transverse confinement of ultrasound



Diffuse: $\langle \rho^2 \rangle = 4Dt$
transition: $\langle \rho^2 \rangle \sim L^2$, not $t^{2/3}$
Localized: $\langle \rho^2 \rangle \sim L\xi$

Slide 53

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bvt4 bart van tiggelen; 09/01/2008

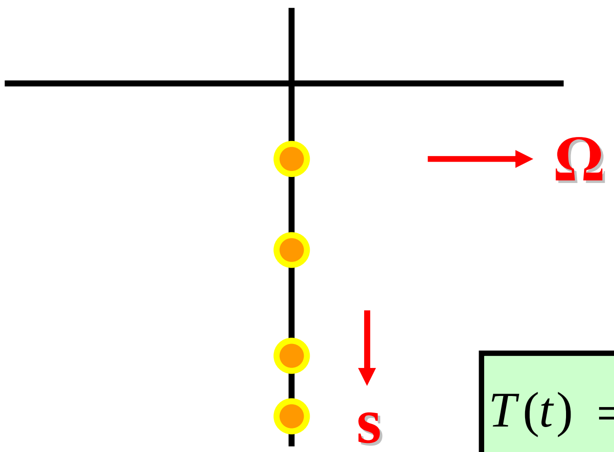
Dynamics of Localization in finite open media

Skipetrov & Van Tiggelen, PRL 2004,2006

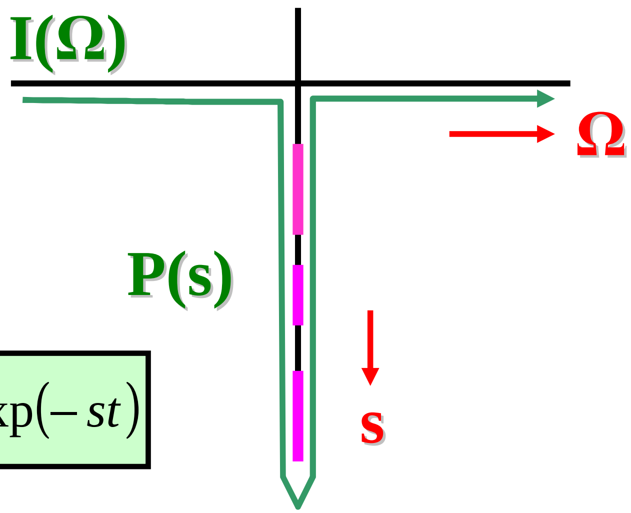
$$-s\,G(z,z',q,s) + \partial_z D(z,s)\,G(z,z',q,s) + q^2 G(z,z',q,s) = \delta(z-z')$$

$$\frac{1}{D(z,s)} = \frac{1}{D_B} + \frac{2}{k^2\ell}\int_{q</3\ell} d^2\mathbf{q}\; G(z,z,q,s)$$

Complex Frequency Ω+is



$$T(t) = \int_0^\infty ds\, P(s)\exp(-st)$$

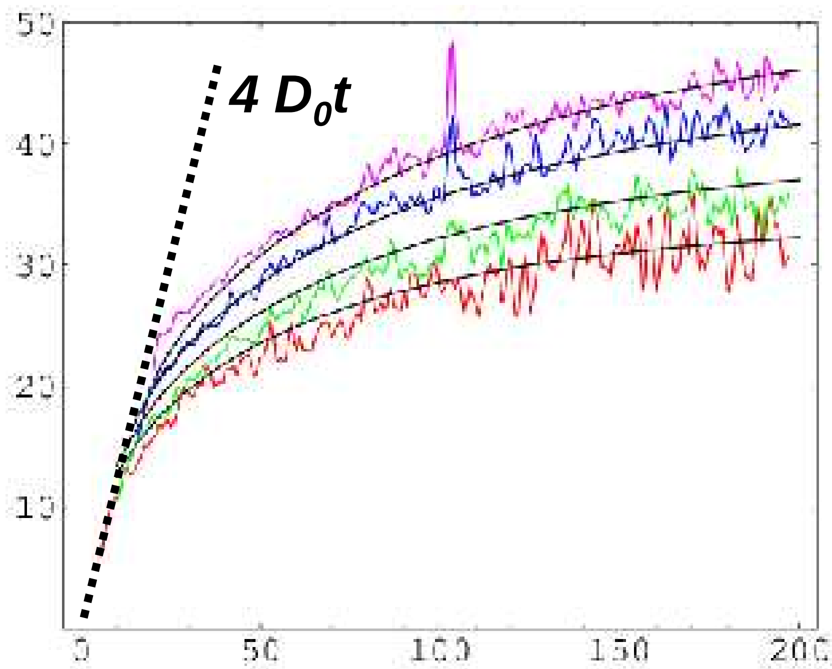


Diffuse regime: simple poles

Localized regime: Riemann sheets

3D Transverse localisation of ultrasound

Transverse size



Time (ms)

ρ = 30 mm

ρ = 25 mm

ρ = 20 mm

ρ = 15 mm

$k\ell \approx 1,82$

v_E=17,4 km/s=3.5 v_p

ξ= 16,3 mm

3D Transverse confinement of ultrasound

$$T(\rho,t) \equiv T(0,t)\exp\left(-\frac{\rho^2}{w(\rho,t)^2}\right)$$

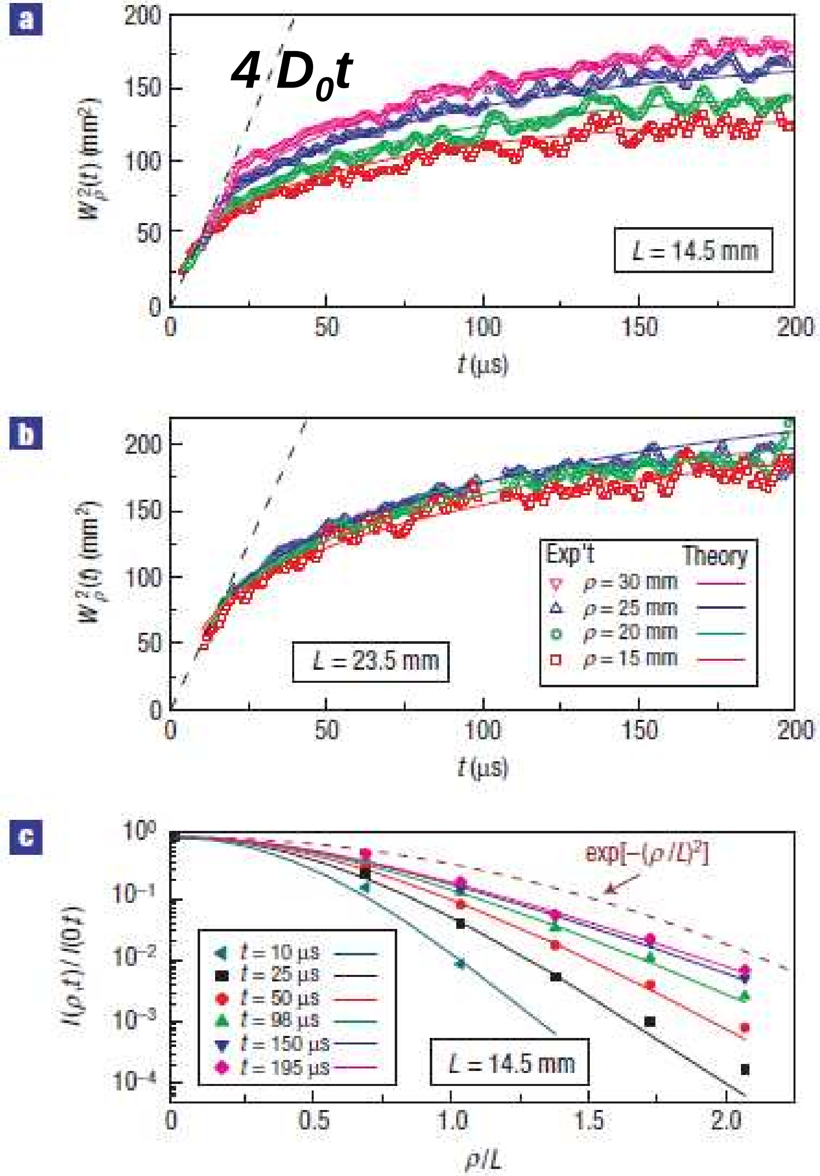
Transverse

Size $w(\rho,t)$

↑

$T(\rho,t)/T(0,t)$

↑



$k\ell \approx 1,82$

v_E=17,4 km/s

=3.5 v_p

ξ= 16,3 mm

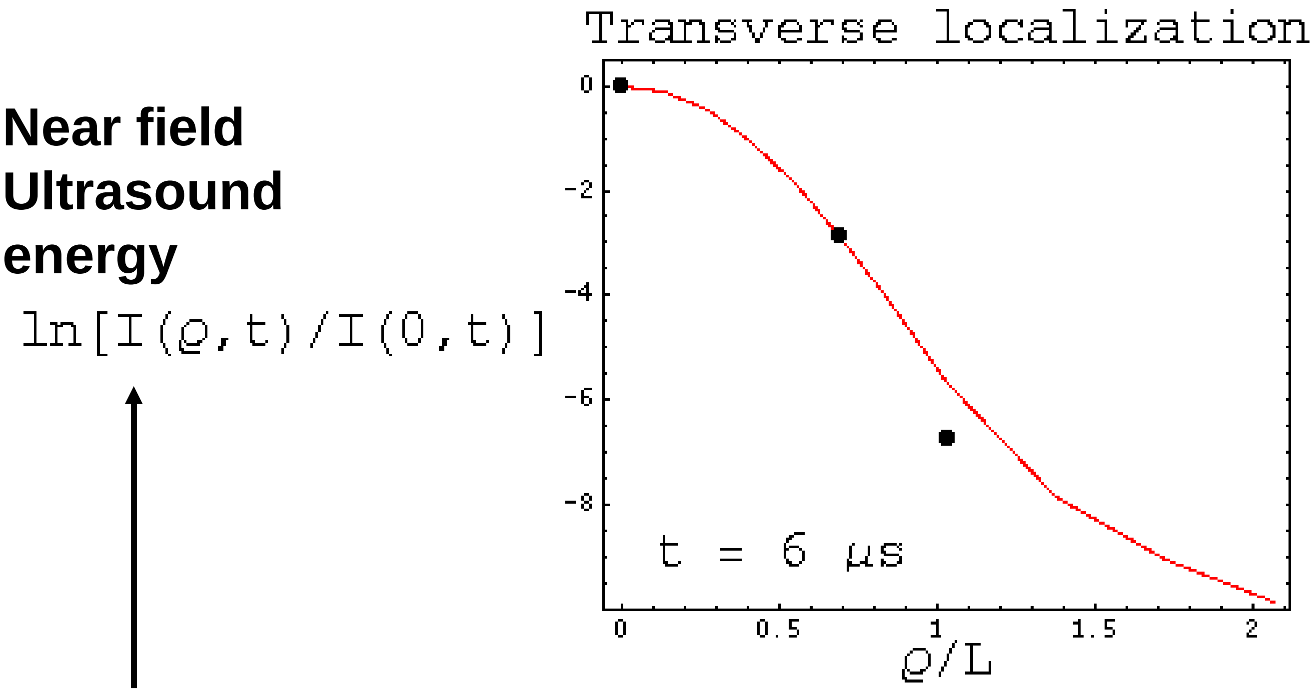
~~D(t)~~

~~D(L)~~

D(r,t-t')

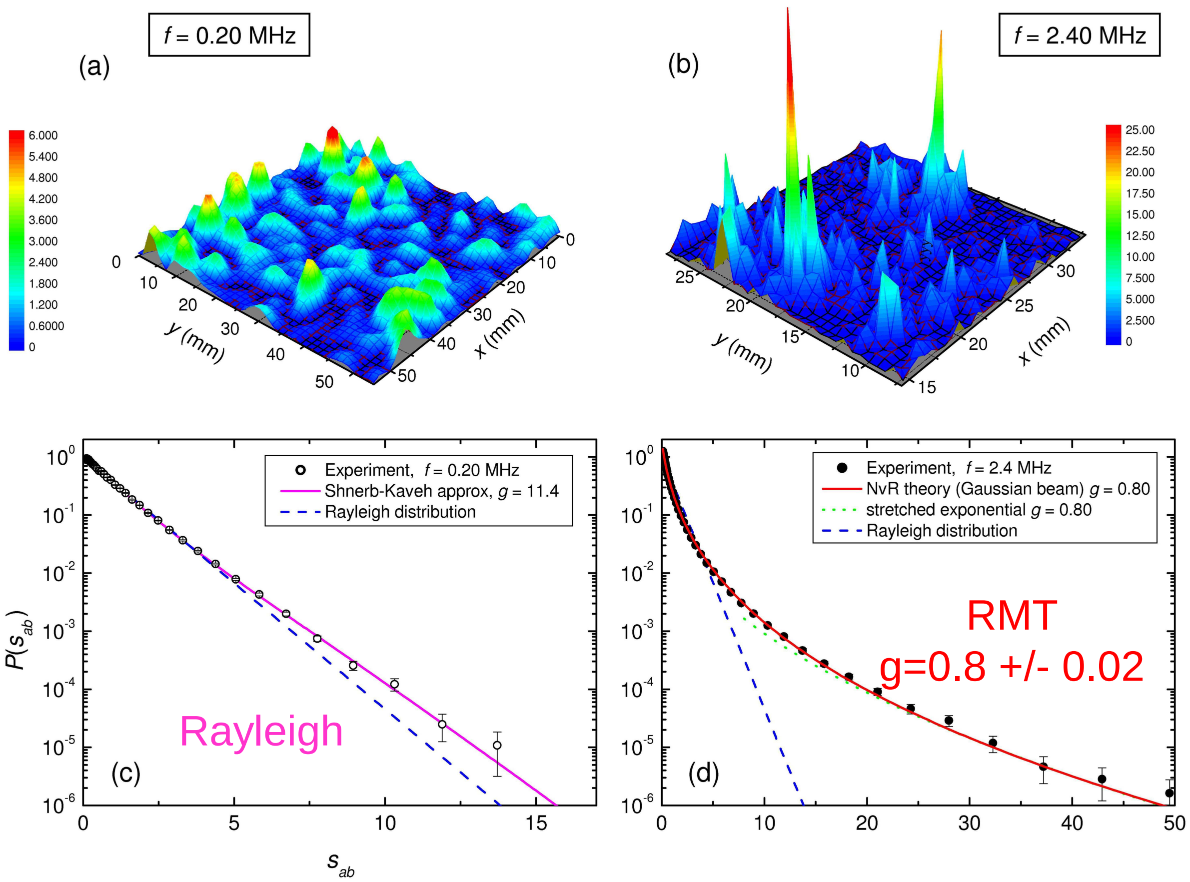
transverse confinement of ultrasound

Near field Ultrasound energy



Transverse position

Speckle distribution of transmission

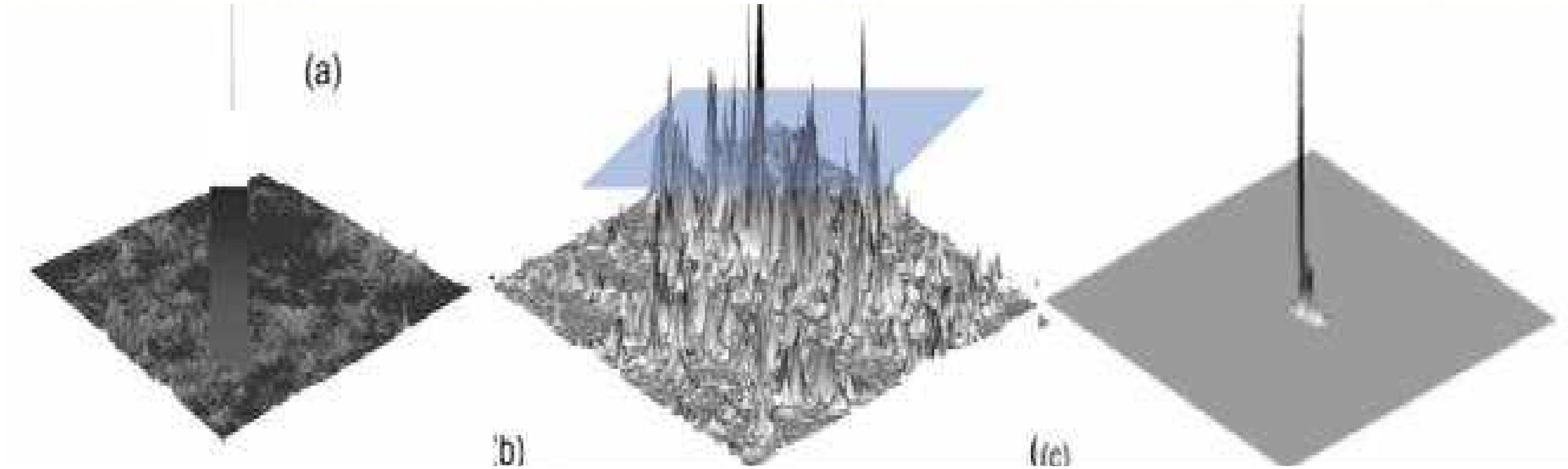


$$P(s_a) = \int_{-i\infty}^{i\infty} \frac{dx}{2\pi i} \exp\left(x s_a - \Phi(x)\right),$$

$$P(s_{ab}) = \int_0^{\infty} \frac{ds_a}{s_a} P(s_a) \exp(-s_{ab}/s_a).$$

$$\Phi(x) = g \ln^2(\sqrt{1 + x/g} + \sqrt{x/g})$$

Multifractality of wave function

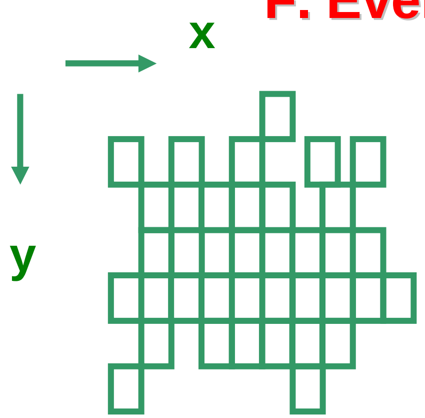


extended critical localized

multifractal

Multifractality of wave function

F. Evers and A. D. Mirlin, Rev. Mod. Phys. 80, 1355 (2008).



Box counting

$$I_b = \frac{\int_{b^d} d^d\mathbf{r}\, I(\mathbf{r})}{\int_{L^d} d^d\mathbf{r}\, I(\mathbf{r})} \qquad \lambda << b << L$$

$$P_q = \sum_b (I_b)^q = \left(\frac{L}{b}\right)^{-d(q-1)+\Delta(q)}$$

Generalized Inverse Participation Ratio

$$P(\log I_b) \propto \left(\frac{L}{b}\right)^{f\left(-\frac{\log I_b}{\log L/b}\right)}$$

Probability Distribution Function

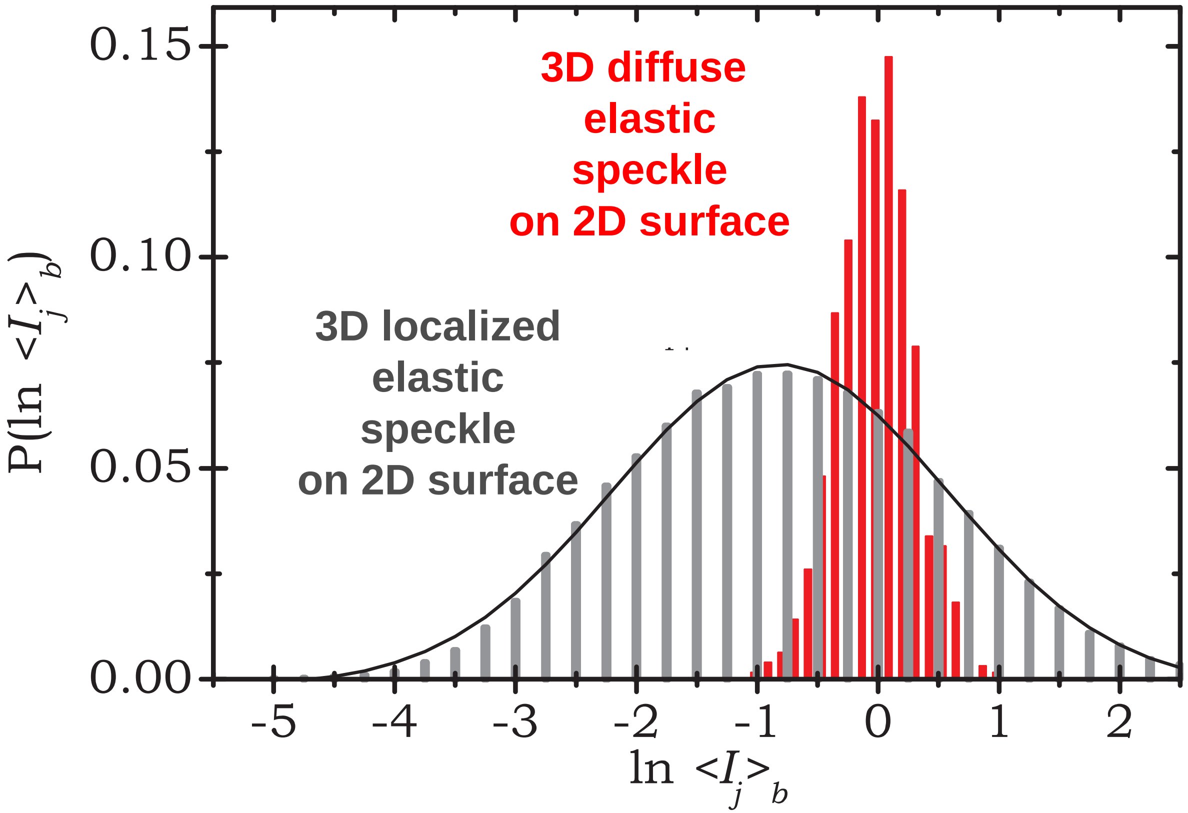
$$\Delta(q) = \gamma q(1-q) \Leftrightarrow f(\alpha) = -\frac{1}{4\gamma}(\alpha - d - \gamma)^2$$

Anomalous gIPR

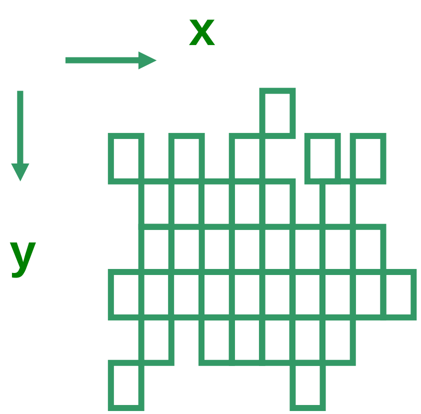
Lognormal PDF

Multifractality of wave function

Faez, Page, Lagendijk and Van Tiggelen (PRL 2010)



Multifractality of wave function



Box counting

$$P(\log I_b) = \frac{1}{N_{L/b}}\left(\frac{b}{L}\right)^{d-f\left(-\frac{\log I_b}{\log L/b}\right)}$$

$$P_q = \sum_b (I_b)^q = \left(\frac{b}{L}\right)^{\tau_q}$$

$$\alpha \equiv \frac{\log I_b}{\log\lambda} \quad \lambda = \frac{b}{L} \downarrow 0$$

$$P_p = N\left\langle I_b^q \right\rangle = \lambda^{-d}\frac{1}{N_\lambda}\int d\log I_b\, I_b^q\, \lambda^{d-f\left(\log I_b / \log\lambda\right)}$$

$$= \frac{\log\lambda}{N_\lambda}\int d\alpha \lambda^{-f(\alpha)+q\alpha} \longrightarrow \begin{cases} \tau_q = \alpha^* q - f(\alpha^*) \\ q = f'(\alpha^*) \end{cases}$$

$$f(\alpha) = f(\alpha^*) + \tfrac{1}{2} f''(\alpha^*)(\alpha-\alpha^*)^2$$

Method of steepest descend

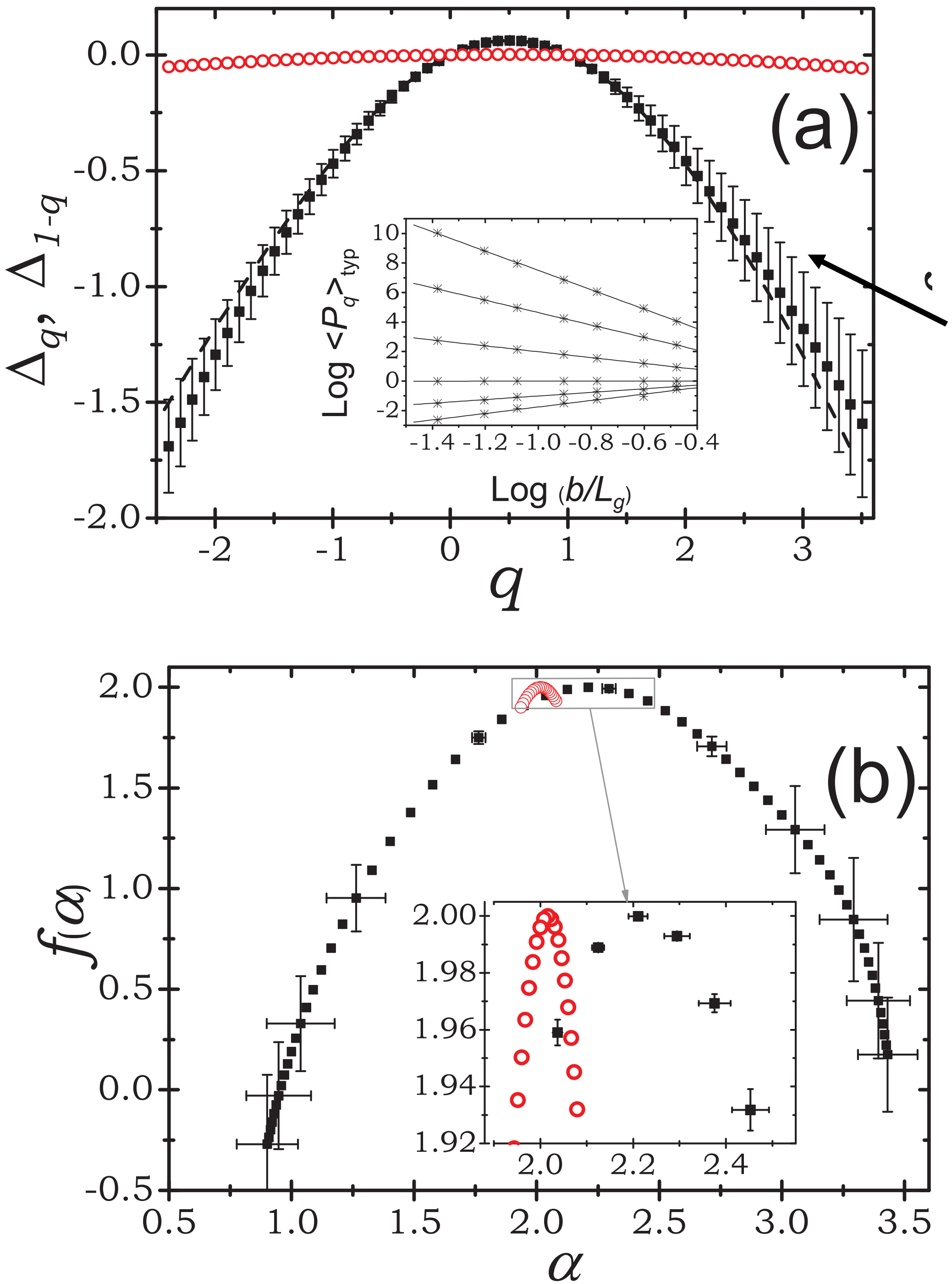
$$\tau_q = \underbrace{-d + dq}_{} + \Delta_q$$

Extended regime:

Anomalous gIPR

$$\Delta(q) = \gamma q(1-q) \Leftrightarrow f(\alpha) = d - \frac{1}{4\gamma}(\alpha - d - \gamma)^2$$

Lognormal PDF



Localized ultrasound

$$\Delta_q \approx 0.21\, q(1-q)$$

is

$$\gamma \propto \frac{1}{g} F(L/\xi(E))$$

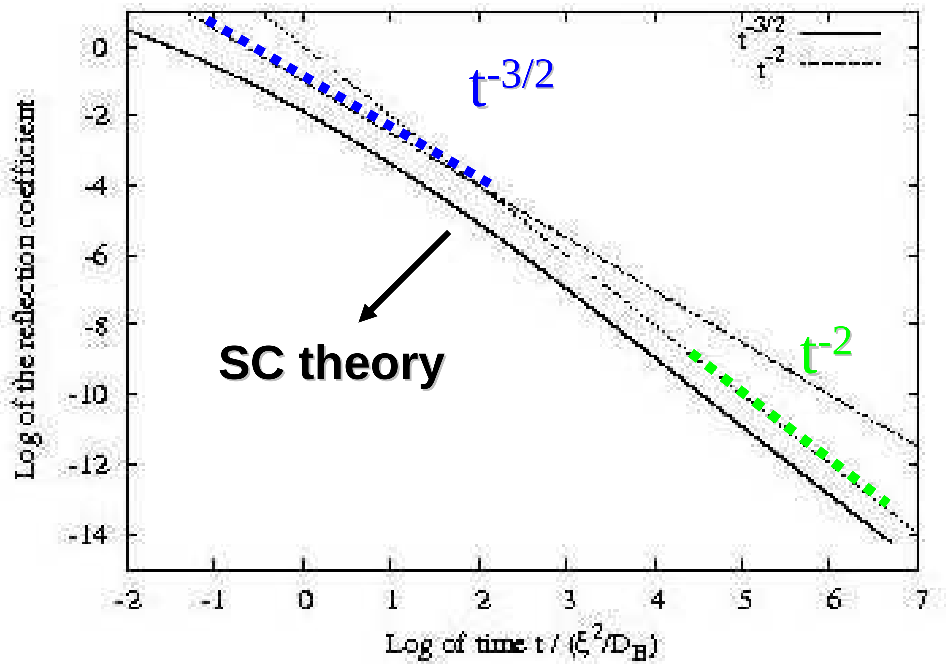
?

VII.d Observation of 3D ultrasound Localization

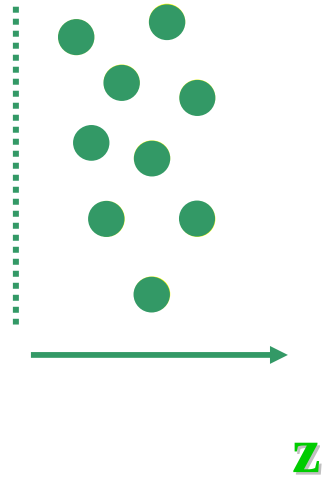
- ✓ Transverse confinement: D(r)!
- ✓ Anomalous dynamic transmission
- ✓ Giant non-poissonian fluctuations g < 1
- ✓ Multifractal wave function
- ? $1/t^2$ reflection (in progress)
- ? Critical exponent
- ? Critical LDOS fluctuations (in progress)

3D, localized half space : kℓ=0.7

R(t)



time/(ζ^2/D_B)

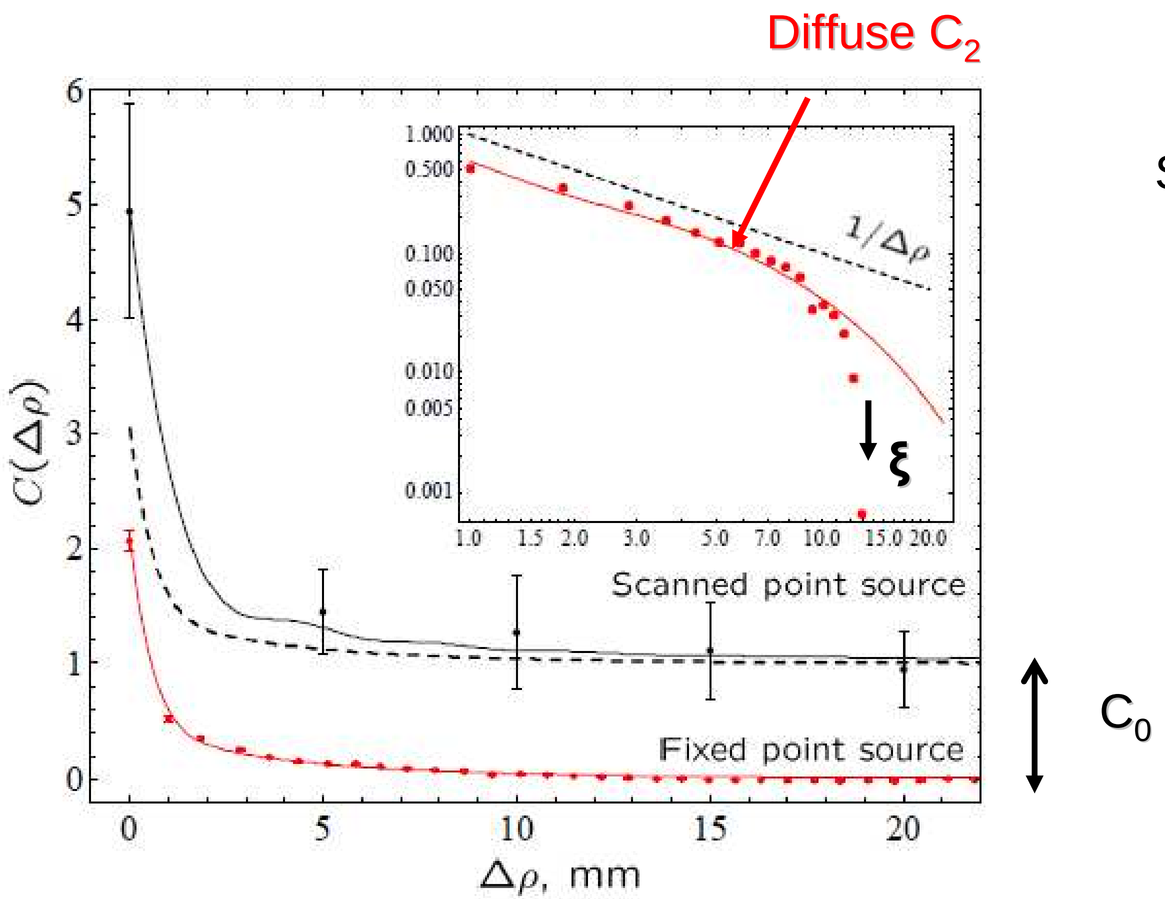


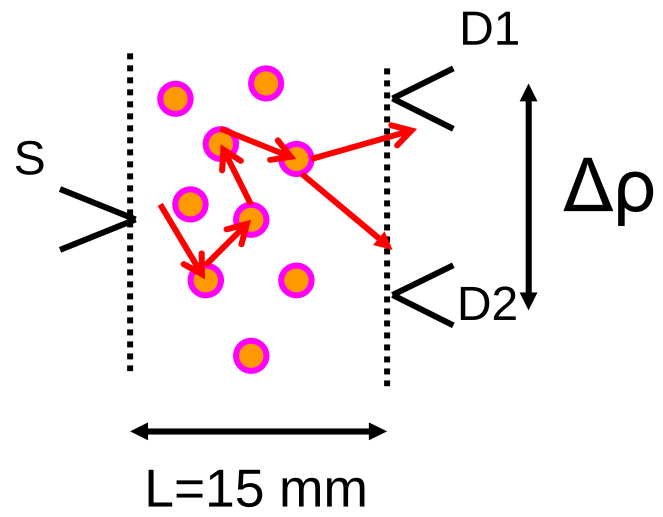
$$R(t) \propto \frac{1}{t^2}$$

1D seismology : Sheng Papanicolaou, 1987

Q1D (DMKP) Titov, Beenakker, 2000

Preliminary observation of infinite range correlations (fluctuations of LDOS)

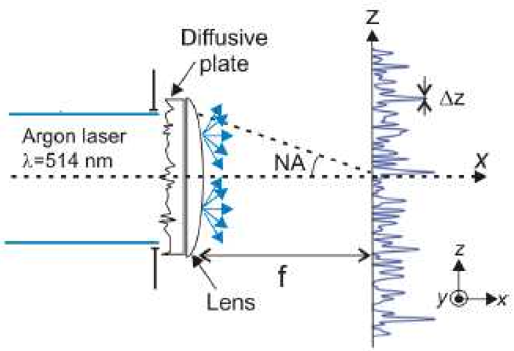




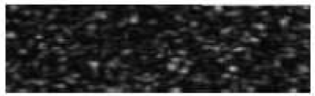
Hildebrand, Page, Skipetrov, Van Tiggelen

IX. Anderson localization of noninteracting cold atoms

Our disordered potential: laser speckle



Laser speckle : diffraction from a rough plate



$$V_{random} \propto \frac{I(r)}{\delta}$$

Blue detuned (atomic transition @ 780 nm)
= repulsive potential

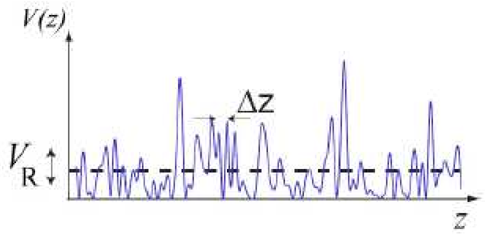
A controlled disorder:

⇒ *Disorder strength* = *laser intensity*

$$\sigma_V = \langle V \rangle \equiv V_R$$

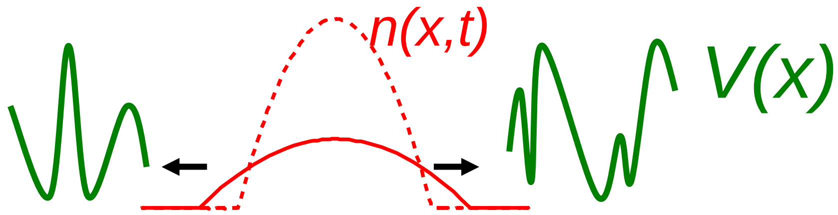
⇒ *Correlation length* = *numerical aperture*

$$\Delta z = \frac{\lambda}{2(N.A.)}$$

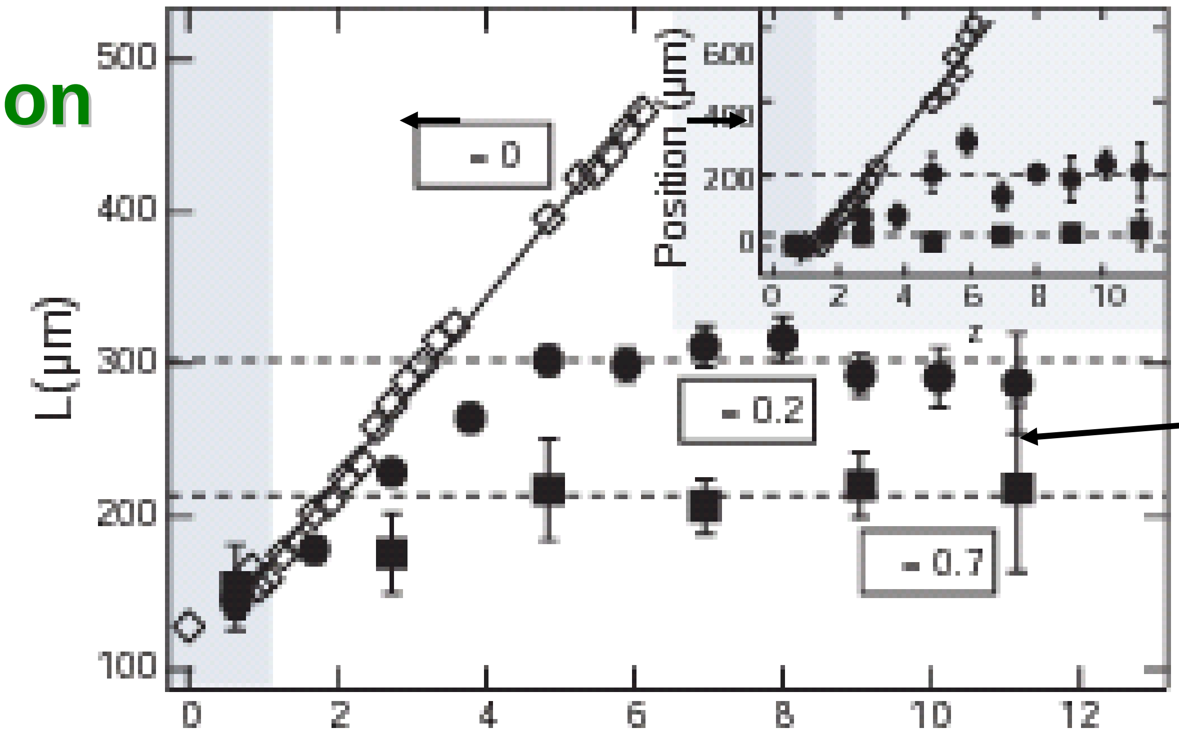


D. Clément et al. NJP 2006

IX.a Q1D BEC in random potential



expansion

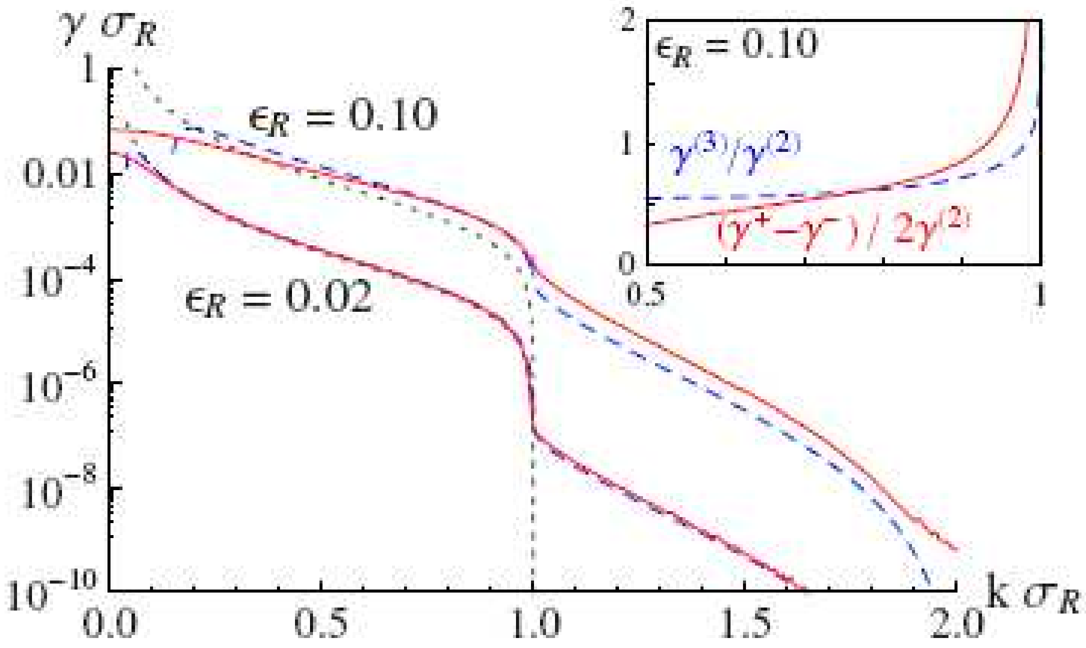


$$\frac{\delta V}{\mu} < 1$$

Time after trap extinction

Palaiseau group, Firenze group
PRL oct 2005

localization in 1D speckle potential



$$V(z) = \sigma_R +$$

$$\gamma^{(n)} = \sigma_R^{-1}\left(\frac{\epsilon_R}{k\sigma_R}\right)^n f_n(k\sigma_R),$$

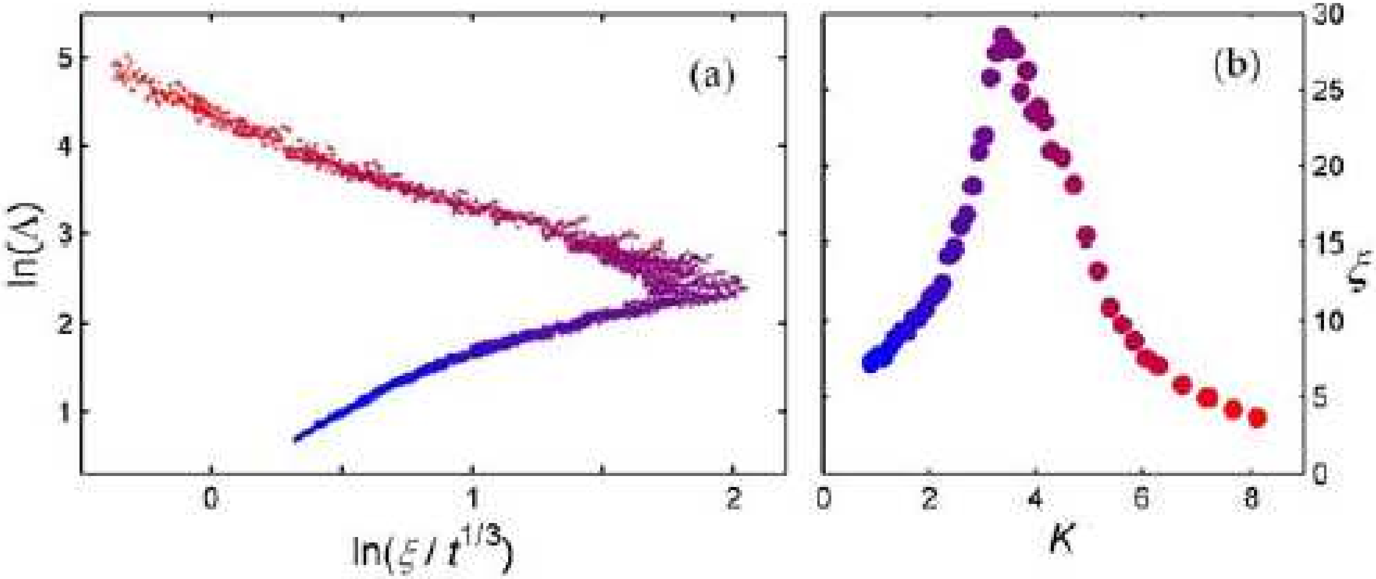
$$T(L) = \exp(-\gamma L)$$

Lugan etal 2009

Experimental verification of localized kicked cold atoms in 3D

Experimental observation of the Anderson transition with atomic matter waves

Julien Chabé[1], Gabriel Lemarié[2], Benoît Grémaud[2], Dominique Delande[2], Pascal Szriftgiser[1] & Jean Claude Garreau[1] PRL, 2008

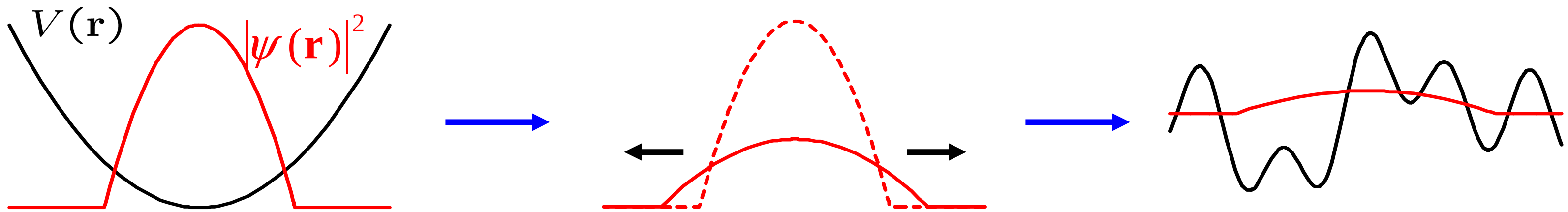


Localization of noninteracting cold atoms in 3D white noise

Skipetrov, Minguzzi, BAvT, Shapiro PRL, 2008

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t)=\left[-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{r})+g\,|\psi(\mathbf{r},t)|^2\right]\psi(\mathbf{r},t)$$

$$\int d^3\mathbf{r}\,|\psi(\mathbf{r},t)|^2=N$$

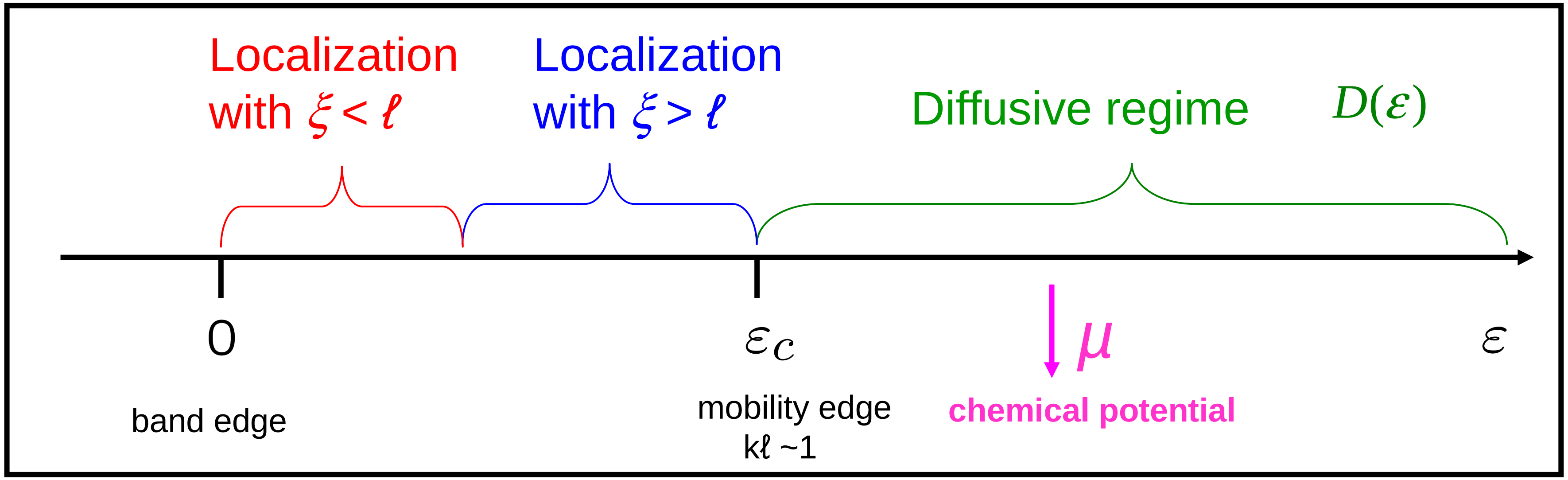


Trap stage — *expansion stage (t=0)* — *Random potential*

$$\psi(\mathbf{r},t)=e^{-i\mu t}\sqrt{\frac{\mu-V(\mathbf{r})}{g}}$$

$$\psi(\mathbf{k},t=0)\propto\theta\left(\mu-\frac{\hbar^2k^2}{2m}\right)$$

$$n(\mathbf{r},t)=|\psi(\mathbf{r},t)|^2=??$$
$$t\rightarrow\infty$$



Average density profile of atoms at large times

Probability of quantum diffusion

$$\langle n(\mathbf{r},t)\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left|\phi_\mu(\mathbf{k})\right|^2 \int_{-\infty}^{\infty} dE\, A(E,\mathbf{k})\, P_E(\mathbf{r},t)$$

Distribution of initial velocities
mv = ħk

Spectral function

$$\frac{1}{\text{volume}} \langle \mathbf{k} | \delta\left(E - \hbar^2 k^2/2m - V(\mathbf{r})\right) | \mathbf{k} \rangle =$$

$$-\frac{1}{\pi} \operatorname{Im} \frac{1}{E - \hbar^2 k^2/2m - \Sigma(E,k)}$$

$$\langle \delta V \rangle = 0 \ ; \langle \delta V(\mathbf{r}) \delta V(\mathbf{r}') \rangle = 4\pi U\, \delta(\mathbf{r} - \mathbf{r}')$$

Mean free path: $\ell = \left(\frac{\hbar^2}{2m}\right)^2 \frac{1}{U} \Longrightarrow E < \left(\frac{2m}{\hbar^2}\right)^3 U^2$ **localized**

Density profile of atoms at large times

Skipetrov, Minguzzi, BAvT, Shapiro PRL, 2008

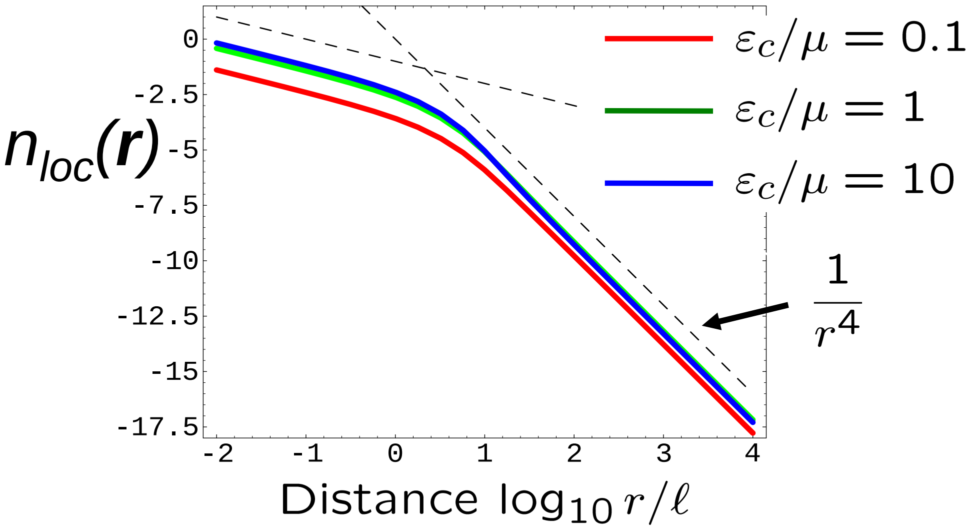
$$n(\mathbf{r},t) = n_{\text{loc}}(\mathbf{r}) + \Delta n_{AD}(\mathbf{r},t)$$

n_{loc} → localized; Δn_{AD} → anomalous diffusion

$$\xi(\varepsilon) \propto \frac{1}{(\varepsilon_c - \varepsilon)^{\nu}} \Rightarrow n_{\text{loc}}(\mathbf{r}) \propto \frac{1}{r^{3+1/\nu}}$$

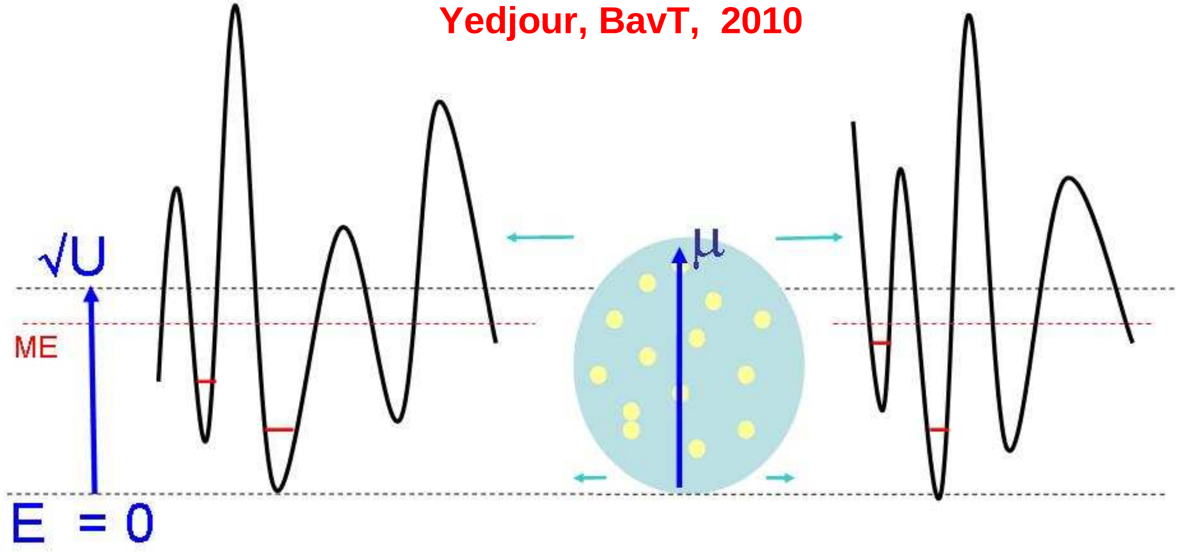
$$D(\varepsilon) \propto (\varepsilon - \varepsilon_c)^{s} \Rightarrow \Delta n_{AD}(\mathbf{r},t) \propto \frac{1}{r^{3-2/s}\, t^{1/s}}$$

Selfconsistent theory (ν=s=1)



Cold atoms in a 3D speckle potential

Kuhn, Miniatura, Delande etal 2007
Yedjour, BavT, 2010



$$\langle V(\mathbf{r})\rangle = \sqrt{U} \qquad \langle \delta V(\mathbf{r})\delta V(\mathbf{r}')\rangle = U\,\mathrm{sinc}^2(\Delta r/\zeta)$$

$$\frac{\hbar}{m\zeta} = 7\,mm/s \qquad \frac{\hbar}{3m} = 600\,\mu m^2/s$$ ← **Mott minimum**

$$\frac{\hbar^2}{2m\zeta^2} \equiv E_\zeta \approx \sqrt{U} \approx \mu = h\times 220\,Hz$$

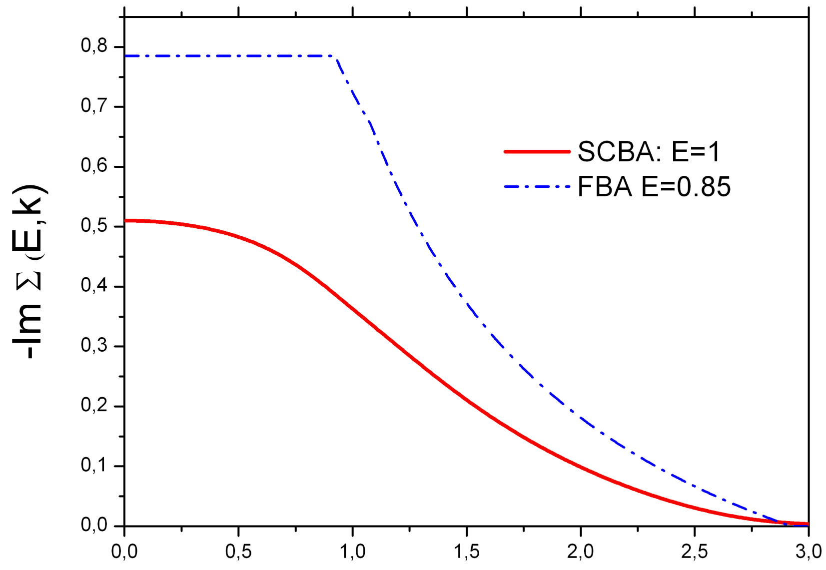
$$\zeta \approx \ell \approx 0.3\,\mu m$$

Self-consistent Born Approximation

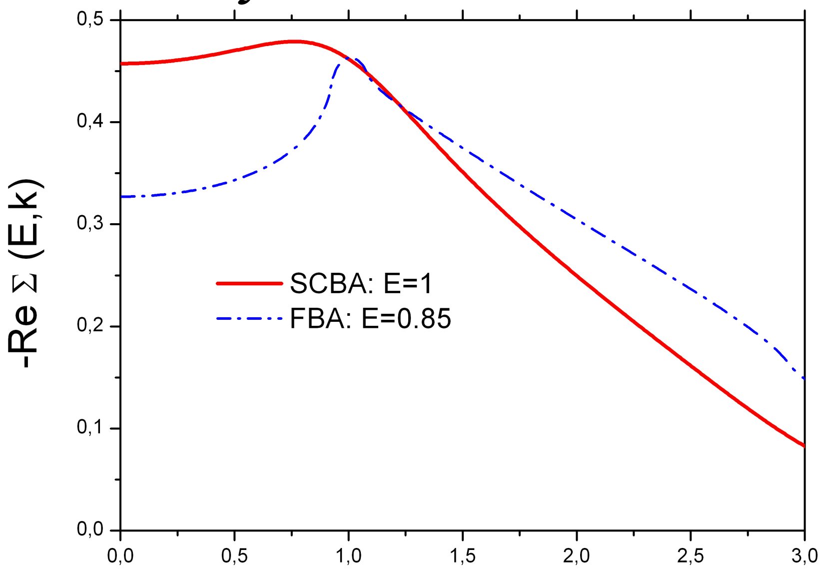
$$G(E,p) = \frac{1}{E - p^2/2m - \sqrt{U} - \Sigma(E,p)}$$

$$\Sigma(E,p) = \sum_{\mathbf{p}'} \frac{S(\mathbf{p}-\mathbf{p}')}{E - p'^2/2m - \sqrt{U} - \Sigma(E,p')}$$

$$E = E_\zeta \quad U = E_\zeta^2$$



p in units of $\hbar/m\zeta$



p in units of $\hbar/m\zeta$

$$D_{\text{Drude}}(E) = \frac{1}{3}\frac{\hbar}{m}\frac{\sum_{\mathbf{p}} p^2\, 2\,\mathrm{Im}^2\, G(E,p)}{\sum_{\mathbf{p}} -\mathrm{Im}\, G(E,p)} = \frac{1}{3}\frac{\hbar}{m} \times \text{"}k\ell\text{"}$$

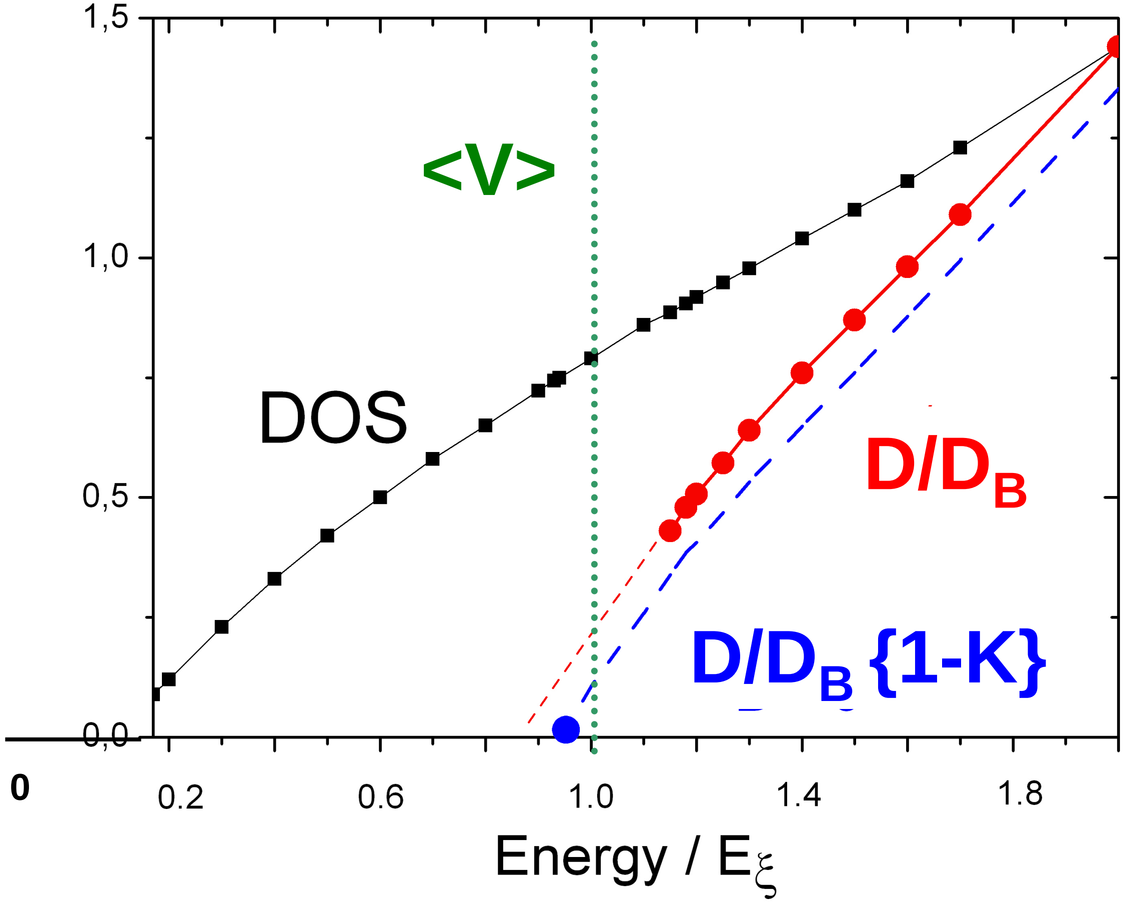
$$D_{\text{Boltz}}(E) = \frac{D_{\text{Drude}}(E)}{1 - \langle \cos\theta \rangle} >> D_{\text{Drude}}(E)$$

Selfconsistent theory of localization

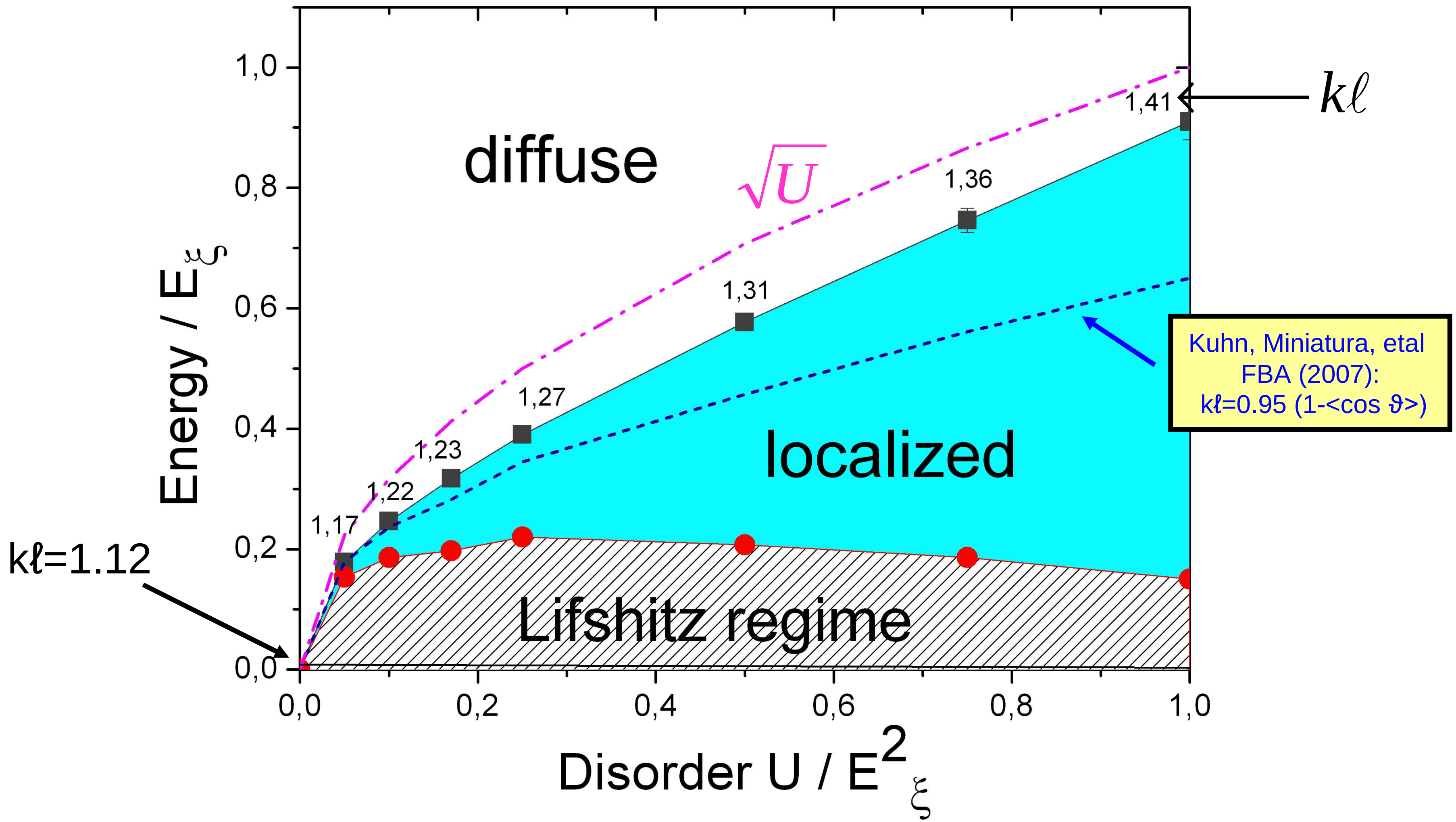
$$D(E) \approx D_{\text{Boltz}}(E)\,\{\,1 - K(E)\,\}$$

$$K(E) = -\frac{6}{\left[\sum_{\mathbf{p}} p^2\, \mathrm{Im}^2 G(E,p)\right]^2} \sum_{\mathbf{p}\mathbf{p}'} \mathrm{Im}^2\, G(E,p')\, \frac{\mathbf{p}\cdot\mathbf{p}'}{(\mathbf{p}+\mathbf{p}')^2}\, \mathrm{Im}^2\, \Sigma(E, \tfrac{1}{2}|\mathbf{p}-\mathbf{p}'|)\, |G(E,p)|^2$$

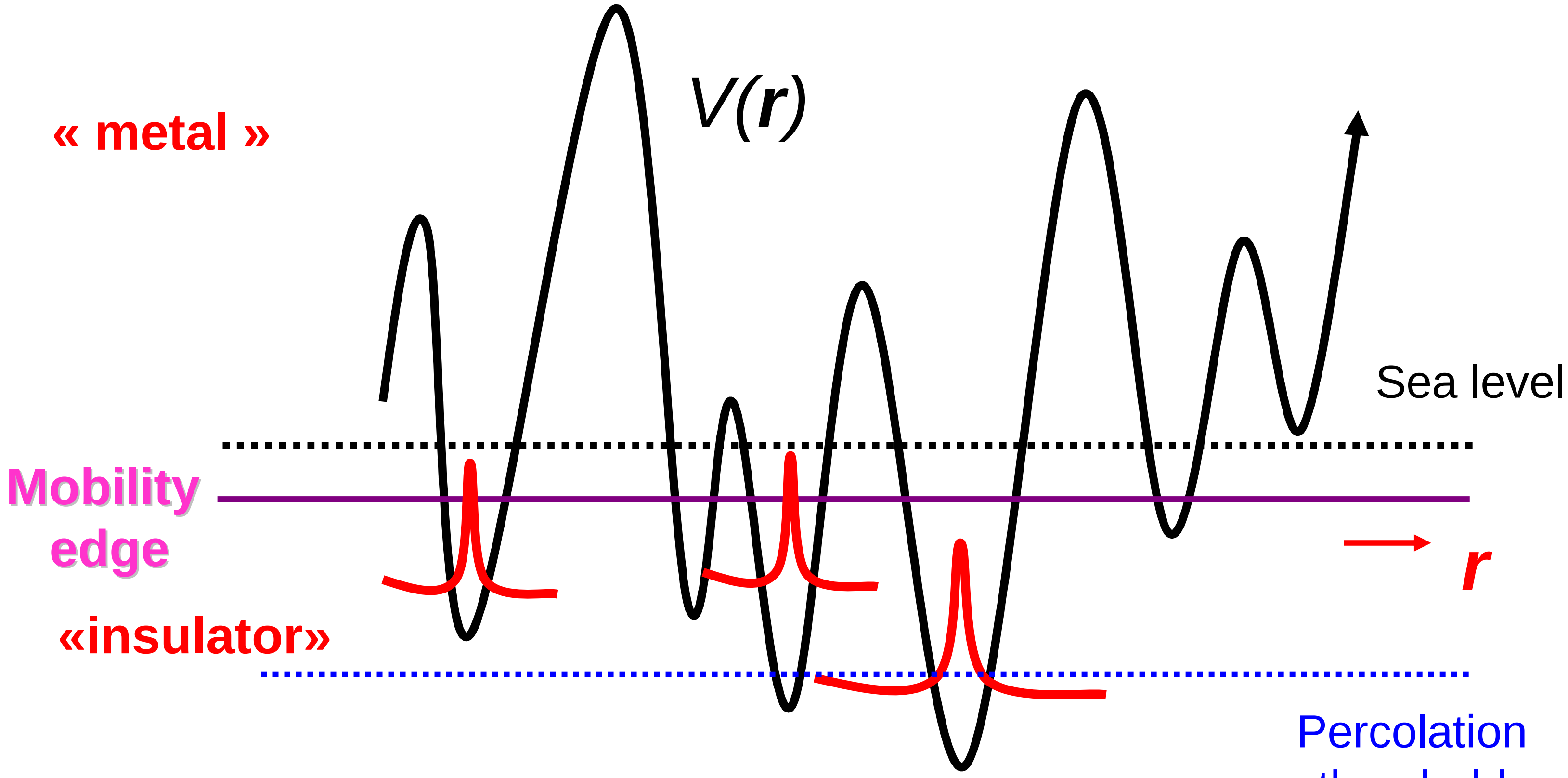
$$U = E_\zeta^2$$



Is 3D cold atoms are delocalized below « sea - level » but above percolation threshold

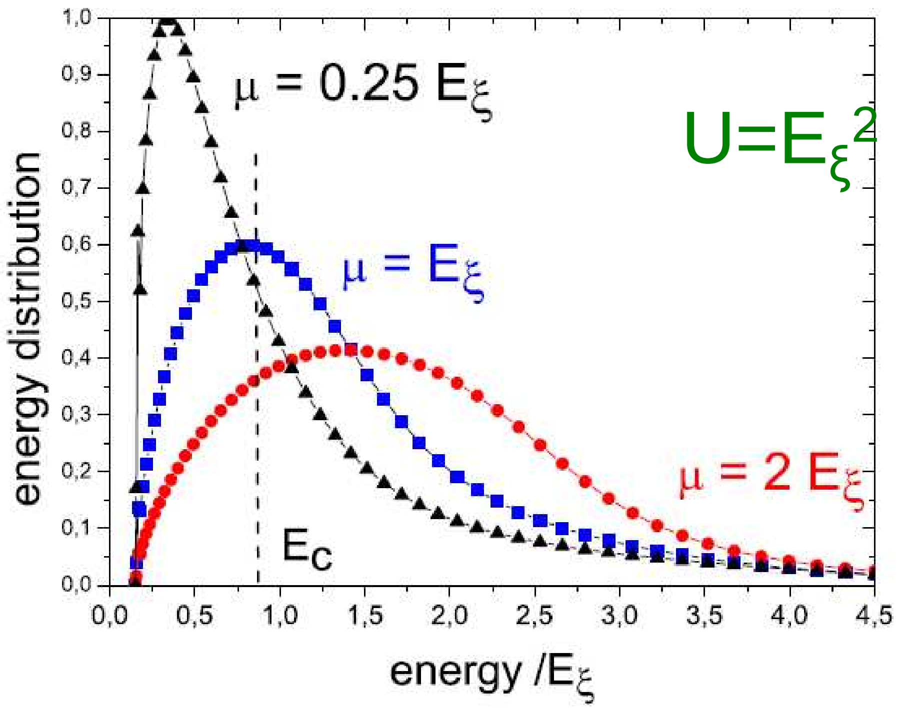


Cold atoms in 3D speckle



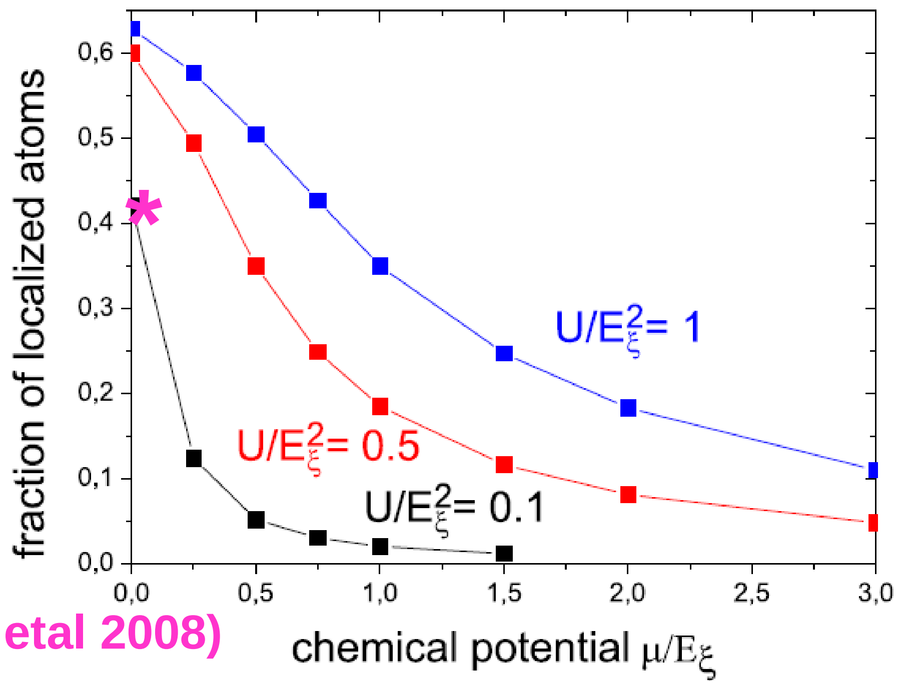
Energy distribution

$$N_\mu(E) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} A(E,k)\left|\phi_\mu(k)\right|^2$$



Fraction of localized atoms

$$f_{loc}(\mu) = \int_0^{E_c} dE\ N_\mu(E)$$



* 45 % in white noise (Skipetrov etal 2008)

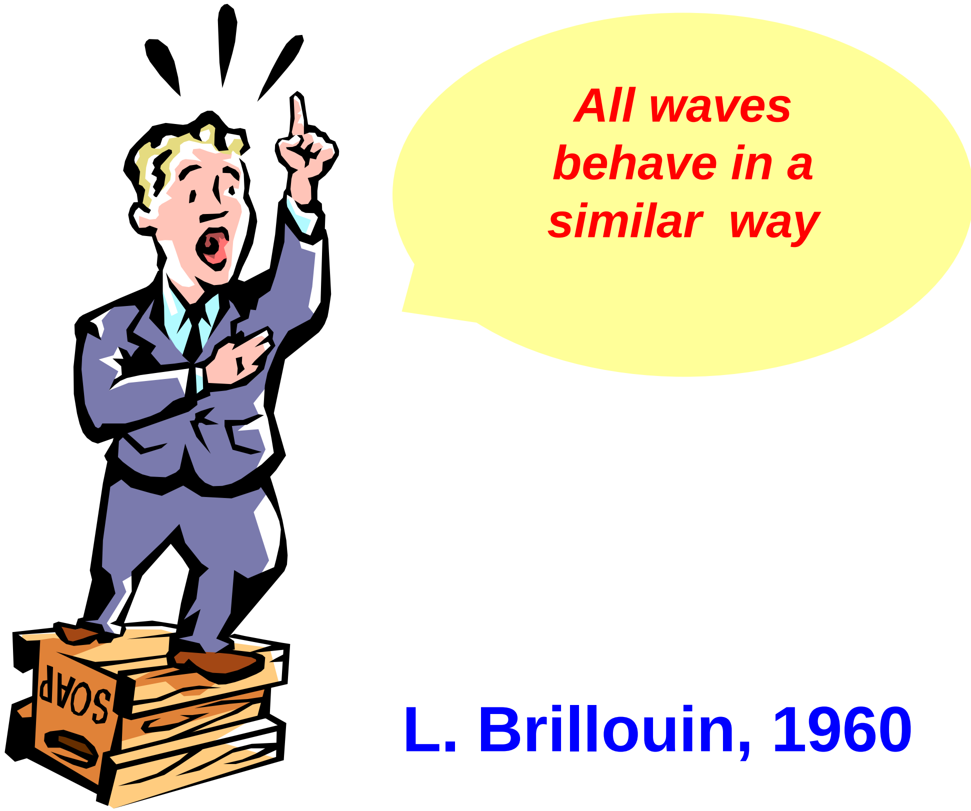
Anderson Localization is still a major theme in condensed matter physics, full of surprises

New experiments (in high dimensions and with « new » waves) exist and are underway.

Thank you for your attention

- **B.A. van Tiggelen, Les Houches 1998**
- **Lagendijk, Van Tiggelen, Wiersman/ Aspect, Inguscio Physics Today, august 2009**
- **C. Müller, D. Delande, Les Houches 2010 http://arxiv.org/abs/1005.0915**

Mesoscopic physics for absolute beginners:



L. Brillouin, 1960