LPMMC, Université Grenoble 1/CNRS
Workshop: Physics and Mathematics of Cold Atoms
PEPS-PTI project - Grenoble, October 11-12, 2012

Bogoliubov Approximation for Random Boson Systems

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Abstract

We investigate the validity of the Bogoliubov c-number approximation in the case of interacting Bose-gas in a homogeneous random media. To take into account a possible occurrence of the **type III** *generalised Bose-Einstein condensation in an infinitesimal band of low* **kinetic-energy modes** without *macroscopic occupation of* **any** *of them, we generalize the c-number substitution procedure in this band of modes with low momenta.*

I.Bogoliubov Theory

1. Zero-Mode Condensate

- Let **interacting** bosons of mass m be enclosed in a *cubic* box $\Lambda = L \times L \times L \subset \mathbb{R}^3$ of the volume $V \equiv |\Lambda| = L^3$, with (*for simplicity*) periodic boundary conditions on $\partial\Lambda$.
- $\varphi(x)$ is two-body potential with Fourier transformation:

$$v(q) = \int_{\mathbb{R}^3} d^3x \; \varphi(x)\, e^{-iqx}, \;\; q \in \mathbb{R}^3.$$

- The second-quantized Hamiltonian acts *in* the boson Fock space $\mathcal{F} := \mathcal{F}_{boson}(\mathcal{L}^2(\Lambda))$, $\Lambda^* := \left\{k \in \mathbb{R}^3 : k_\alpha = \frac{2\pi n_\alpha}{L}, n_\alpha \in \mathbb{Z}, \alpha = 1,2,3\right\}$

$$H_\Lambda = \sum_{k\in\Lambda^*} \varepsilon_k a_k^* a_k + \frac{1}{2V} \sum_{k_1,k_2,q\in\Lambda^*} v(q)\, a^*_{k_1+q} a^*_{k_2-q} a_{k_1} a_{k_2} \;.$$

• Here $\varepsilon_k = \hbar^2 k^2/2m$ is the one-particle excitations spectrum.
• $a_k^{\#} := \left\{a_k^*, a_k\right\}$ are CCR boson creation and annihilation operators in the one-particle *kinetic-energy* eigenvectors

$$\psi_k(x) = \frac{1}{\sqrt{V}}\, e^{ikx} \in \mathcal{L}^2(\Lambda),\ \ k \in \Lambda^*,$$

$$a_k := a(\psi_k) = \int_\Lambda dx\ \overline{\psi_k}(x)\, a(x)\ \ .$$

• $a^{\#}(x)$ are boson-field operators in the Fock space over $\mathcal{L}^2(\Lambda)$.
• Below we **suppose** that two-body interaction potential is:
(A) $\varphi(x) = \varphi(\|x\|)$ (*isotropic*) and $\varphi \in \mathcal{L}^1(\mathbb{R}^3)$ (*absolutely integrable* functions)
(B) $v(k)$ is a (*real continuous*) function, satisfying $v(0) > 0$ and $0 \le v(k) \le v(0)$ for $k \in \mathbb{R}^3$.

2. The First Ansatz: the Bogoliubov Hamiltonian H_Λ^B

- If one expects that Bose–Einstein condensation (BEC), which occurs in the mode $k = 0$ for the Perfect Bose–Gas (PBG), **persists** for a weak interaction $\varphi(x)$, then Bogoliubov proposed to **truncate** H_Λ and to keep only the *most important* terms, in which at least **two** operators a_0^*, a_0 are involved.
- This is the Bogoliubov Weakly Imperfect Bose-Gas (WIBG):

$$H_\Lambda^B := T_\Lambda + U_\Lambda^D + U_\Lambda^{ND} + \frac{v(0)}{2V} a_0^{*2} a_0^2 \ ,$$

$$U_\Lambda^D := \frac{v(0)}{V} a_0^* a_0 \sum_{k\in\Lambda^*, k\neq 0} a_k^* a_k + \sum_{k\in\Lambda^*\setminus\{0\}} \frac{v(k)}{2} \frac{a_0^* a_0}{V} (a_k^* a_k + a_{-k}^* a_{-k}),$$

$$U_\Lambda^{ND} := \sum_{k\in\Lambda^*\setminus\{0\}} \frac{v(k)}{2} \left(a_k^* a_{-k}^* \frac{a_0^2}{V} + \frac{a_0^{*2}}{V} a_k a_{-k} \right) \ .$$

3. The Second Ansatz: the c-Number Approximation

• For the periodic boundary conditions on $\partial\Lambda$, let $\mathcal{F}_0 := \mathcal{F}_{boson}(\mathcal{H}_0)$ be the boson Fock space constructed on the one-dimensional Hilbert space $\mathcal{H}_0$ spanned by $\psi_{k=0}(x) = 1/\sqrt{V}$.

• Let $\mathcal{F}_0' := \mathcal{F}_{boson}\left(\mathcal{H}_0^{\perp}\right)$ be the Fock space constructed on the orthogonal complement $\mathcal{H}_0^{\perp}$. Then $\mathcal{F}_{boson}(\mathcal{H} = \mathcal{L}^2(\Lambda)) \equiv \mathcal{F}_{boson}(\mathcal{H}_0 \oplus \mathcal{H}_0^{\perp})$ is isomorphic to the *tensor product*:

$$\mathcal{F}_{boson}\left(\mathcal{H}_0 \oplus \mathcal{H}_0^{\perp}\right) \approx \mathcal{F}_{boson}(\mathcal{H}_0) \otimes \mathcal{F}_{boson}(\mathcal{H}_0^{\perp}),$$

• For any complex number $c \in \mathbb{C}$ the coherent vector in $\mathcal{F}_0$ is

$$\psi_{0\Lambda}(c) := e^{-V|c|^2/2} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sqrt{V} c\right)^k (a_0^*)^k \, \Omega_0 = e^{(-V|c|^2/2 + \sqrt{V} c \, a_0^*)} \, \Omega_0 \, ,$$

where Ω_0 is the vacuum of $\mathcal{F}$. Notice that

$$\frac{a_0}{\sqrt{V}} \, \psi_{0\Lambda}(c) = \mathbf{c} \; \psi_{0\Lambda}(c) \equiv \mathbf{c} \cdot \mathbf{I} \; \psi_{0\Lambda}(c) \, .$$

• **Definition.** The *c-number* Bogoliubov approximation of the grand-canonical Hamiltonian ($N_\Lambda := \sum_{k\in\Lambda^*} a_k^* a_k := a_0^* a_0 + N'_\Lambda$):

$H_\Lambda^B(\mu) := H_\Lambda^B - \mu N_\Lambda$, dom$(H_\Lambda^B(\mu)) \subset \mathcal{F} \approx \mathcal{F}_{boson}(\mathcal{H}_0) \otimes \mathcal{F}_{boson}(\mathcal{H}_0^\perp)$

is a *self-adjoint operator* $H_\Lambda^B(c,\mu)$ defined in $\mathcal{F}'_0 = \mathcal{F}_{boson}\left(\mathcal{H}_0^\perp\right)$, for any fixed vector $\psi_{0\Lambda}(c)$, by the sesquilinear form:

$$\left(\psi'_1, H_\Lambda^B(c,\mu)\psi'_2\right)_{\mathcal{F}'_0} := \left(\psi_{0\Lambda}(c) \otimes \psi'_1, H_\Lambda^B(\mu)\,\psi_{0\Lambda}(c) \otimes \psi'_2\right)_{\mathcal{F}} ,$$

for vectors $(\psi_{0\Lambda}(c) \otimes \psi'_{1,2}) \in$ *form-domain* of the operator $H_\Lambda^B(\mu)$.

• **Remark.** Since $(a_0/\sqrt{V})\ \psi_{0\Lambda}(c) = c \cdot \mathbf{I}\ \psi_{0\Lambda}(c)$, the *c-number* approximation is **equivalent** to *substitutions*:

$$a_0/\sqrt{V} \to c \cdot \mathbf{I}\ ,\ a_0^*/\sqrt{V} \to c^* \cdot \mathbf{I}$$

in the Bogoliubov truncated Hamiltonian
$H_\Lambda^B(\mu) \to H_\Lambda^B(c,\mu) =: H'_\Lambda(z) - \mu(|z|^2\,\mathbf{I} + N'_\Lambda)$, $z := c\,\sqrt{V}$.

3. The Third Ansatz: the Bogoliubov Spectrum

• Since Hamiltonian $H_{\Lambda}^{B}(c,\mu)$ is a bilinear form in boson operators $\{a_k^{\sharp}\}_{k\in\Lambda^*\setminus\{0\}}$, one can diagonalise it by the Bogoliubov *canonical* $u-v$ transformation to new boson operators $\{b_k^{\sharp}\}_{k\in\Lambda^*\setminus\{0\}}$

$$\mathcal{H}_{\Lambda}^{B}\left(c^{\#},\mu\right) = \sum_{k\in\Lambda^*\setminus\{0\}} E_k^B b_k^* b_k + \frac{1}{2}\sum_{k\in\Lambda^*\setminus\{0\}} \left(E_k^B - f_k\right)$$

$$-V(\mu\,|c|^2 - \frac{v(0)}{2}|c|^4)\ ,\ \ f_k = \varepsilon_k - \mu + |c|^2\left[v\left(0\right) + v\left(k\right)\right],$$

$$E_k^B = \sqrt{\left(\varepsilon_k - \mu + |c|^2 v(0)\right)\left(\varepsilon_k - \mu + |c|^2 v(0) + 2\,|c|^2 v(k)\right)}$$

• If one fixes the parameter c by equation $\mu = |c|^2 v\left(0\right)$, then the Bogoliubov spectrum E_k^B is **gapless** and verifies the Landau criterium of **superfluidity** in the presence of the condensate $\rho_{cond} = |c|^2 > 0$. There is NO superfluidity without condensation.

NB Let the initial energy of the fluid moving in a capillary with velocity v be: $\mathcal{E}_{\mathsf{v}} = \mathcal{E}_0 + M\mathsf{v}^2/2$.

• **Ansatz:** Any interaction of the quantum liquid with *external world* manifests as elementary excitations $E_k^B \geq 0$, the *friction* have to produce them, *decreasing* the initial energy to $\mathcal{E}_{\mathsf{v}'}$:

$\mathcal{E}_{\mathsf{v}} = \mathcal{E}_{\mathsf{v}'} + E_k^B \ , \ M\mathsf{v} = M\mathsf{v}' + \hbar k \ \Rightarrow \ \frac{1}{2}\hbar k\,(\mathsf{v} + \mathsf{v}') = E_k^B \geq 0 \ \Rightarrow$

$$[\text{Landau}(1947)]: \ \ \mathsf{v} > \frac{E_k^B}{\hbar\ k} > \mathsf{v}' \ \Rightarrow \ \mathsf{v} > \inf_k \left[\frac{E_k^B}{\hbar\ k}\right] =: \mathsf{v}_{crit} > 0 \ .$$

• **Bogoliubov spectrum** of elementary excitations in the Weakly Imperfect Bose-Gas in the presence of the condensate ρ_{cond}:

$$E_k^B = \sqrt{\varepsilon_k\{\varepsilon_k + 2v(k)\rho_{cond}\}} \geq \hbar\, k\, \mathsf{v}_{crit}(\rho_{cond}) > 0 \Leftrightarrow \rho_{cond} > 0,$$

$\varepsilon_k = \hbar^2 k^2/2m$ (spectrum of the free bosons), $v(k) \leq v(0)$ (Fourier transformation of the two-body interaction potential)

II. How Fragile is the One-Mode Bogoliubov Theory ?
1. Generalised (or Fragmented) Condensation

- Let us modify the kinetic-energy operator T_Λ by a *certain fraction* of the *total* two-body interaction called the *forward scattering*.
- The Hamiltonian and the Grand Partition Function have the following form:

$$H_\Lambda^I = \sum_{k\in\Lambda^*} \varepsilon_k a_k^* a_k + \frac{1}{2V}\sum_{k\in\Lambda^*} v(0) a_k^* a_k^* a_k a_k, \quad v(q=0) > 0,$$

$$\Xi_\Lambda^I(\beta,\mu) = Tr_{\mathcal{F}_B} e^{-\beta\left(H_\Lambda^I - \mu N_\Lambda\right)} = \prod_{k\in\Lambda^*}\sum_{n_k=0}^{\infty} e^{-\beta\left[(\varepsilon_k-\mu)n_k + \frac{v(0)}{2V}(n_k^2 - n_k)\right]} .$$

$$p_\Lambda\left[H_\Lambda^I\right] = \frac{1}{\beta V}\ln \Xi_\Lambda^I(\beta,\mu) .$$

• Since $H^I_\Lambda := T_\Lambda + U^{v(0)}_\Lambda \geq T_\Lambda$, we can establish for the pressure the estimates from above and from below: ,

$$p_\Lambda [T_\Lambda] \geq p_\Lambda \left[H^I_\Lambda\right] \geq$$
$$\frac{1}{\beta V} \sum_{k\in\Lambda^*} \ln \sum_{n_k=0}^{[\ln V]} e^{-\beta\left[(\varepsilon_k-\mu)n_k+v(0)(n_k^2-n_k)/2V\right]} \geq$$
$$\frac{1}{\beta V} \sum_{k\in\Lambda^*} \ln \left\{ e^{-\beta v(0)[\ln V]^2/2V} \; \frac{1-e^{-\beta(\varepsilon_k-\mu)([\ln V]-1)}}{1-e^{-\beta(\varepsilon_k-\mu)}} \right\} \Big|_{V\to\infty} =$$
$$= p_\Lambda [T_\Lambda] \; .$$

• This yields coincidence of pressures with and without forward scattering interaction.

- **Theorem 1.4** $\lim_{\Lambda} p_{\Lambda}[T_{\Lambda}] = \lim_{\Lambda} p_{\Lambda}\left[H_{\Lambda}^{I}\right]$.
- By the convexity inequality one obtains:

$$p_{\Lambda}[T_{\Lambda}] - p_{\Lambda}\left[H_{\Lambda}^{I}\right] \geq \frac{v(0)}{2}\left\{\frac{1}{V^2}\sum_{k\in\Lambda^*}\left(\left\langle N_k^2\right\rangle_{H_{\Lambda}^{I}} - \langle N_k\rangle_{H_{\Lambda}^{I}}\right)\right\}.$$

- Since for the Gibbs state $\langle - \rangle_{H_{\Lambda}^{I}}$ one has

$$\left|\langle A^* B\rangle_{H_{\Lambda}^{I}}\right|^2 \leq \langle A^* A\rangle_{H_{\Lambda}^{I}}\langle BB^*\rangle_{H_{\Lambda}^{I}} \;\Rightarrow\; (\langle N_k\rangle_{H_{\Lambda}^{I}}/V)^2 \leq \left\langle N_k^2\right\rangle_{H_{\Lambda}^{I}}/V^2.$$

Theorem 1.4 and the convexity inequality imply:

$$\lim_{\Lambda}\frac{1}{V}\langle N_k\rangle_{H_{\Lambda}^{I}} = 0 \quad k \in \{\Lambda^*\} \ ,$$

i.e. NO condensation in ANY single mode = type III generalised condensation à la van den Berg-Lewis-Pulé.

- Does BEC exist in the model H^I_Λ ?

YES: By Theorem 1.4 and by the Griffiths lemma one has

$$\rho_{c,I}(\beta) = \rho_c(\beta) < \infty \ ,$$

because

$$\rho_I(\beta, \mu < 0) := \lim_{\Lambda} \partial_\mu p_\Lambda \left[H^I_\Lambda\right] = \lim_{\Lambda} \partial_\mu p_\Lambda \left[T_\Lambda\right] = \rho(\beta, \mu < 0).$$

- Generalised condensation à la van den Berg-Lewis-Pulé is

$$\lim_{\delta \downarrow 0} \lim_{\Lambda} \sum_{k:\varepsilon_k \leqslant \delta} \frac{1}{V} \langle (N_\Lambda(\psi_k)\rangle_{H^I_\Lambda} = (\rho - \rho_c(\beta))_+ \ .$$

2. Random Homogeneous (Ergodic) External Potentials. Can we save the Bogoliubov Theory (BT)?

2.1 Random Eigenfunctions/Kinetic-Energy Eigenfunctions

- Recall that for the random Schrödinger operator in $\Lambda \subset \mathbb{R}^d$:

$$h_\Lambda^\omega \, \phi_j^\omega = (t_\Lambda + v^\omega)_\Lambda \, \phi_j^\omega = E_j^\omega \, \phi_j^\omega \ , \text{ for a.a. } \omega \in \Omega \ .$$

- Let $N_\Lambda(\phi_j^\omega)$ be particle-number operator in the eigenstate ϕ_j^ω.

$$N_\Lambda := \sum_{j \geq 1} N_\Lambda(\phi_j^\omega) := \sum_{j \geq 1} a^*(\phi_j^\omega) a(\phi_j^\omega)$$

is the *total* number operator in the boson Fock space $\mathfrak{F}_B(L^2(\Lambda))$, where $a(\phi_j^\omega) := \int_\Lambda dx \, \overline{\phi_j^\omega}(x) \, a(x)$, and $\{\phi_j^\omega\}_{j \geq 1}$ is a basis in $L^2(\Lambda)$.

- Let $t_\Lambda \, \psi_k = \varepsilon_k \, \psi_k$ be the kinetic-energy operator, eigenfunctions and eigenvalues $\varepsilon_k = \hbar^2 \, k^2 / 2m$. One of the **key hypothesis** of the **BT** is the ground-state (or zero-mode $\psi_{k=0}$) condensation.

2.2 The First Main Theorem

- **Theorem 2.1**[Jaeck-Pulé-Zagrebnov]

Let $H_\Lambda^\omega := T_\Lambda + V_\Lambda^\omega + U_\Lambda$ be *many-body* Hamiltonians of interacting bosons in random external potential (trap) V_Λ^ω. If particle *interaction* U_Λ commutes with **any** of the operators $N_\Lambda(\phi_j^\omega)$ (*local gauge invariance*), then

$$\lim_{\delta\downarrow 0}\lim_{\Lambda}\sum_{j:E_j^\omega\leqslant\delta}\frac{1}{V}\langle(N_\Lambda(\phi_j^\omega))\rangle_{H_\Lambda^\omega} > 0 \Leftrightarrow$$

$$\Leftrightarrow \lim_{\gamma\downarrow 0}\liminf_{\Lambda}\sum_{k:\varepsilon_k\leqslant\gamma}\frac{1}{V}\langle N_\Lambda(\psi_k)\rangle_{H_\Lambda^\omega} > 0 \ ,$$

and: $\lim_{\gamma\downarrow 0}\lim_\Lambda\sum_{k:\varepsilon_k>\gamma}\langle N_\Lambda(\psi_k)\rangle_{H_\Lambda^\omega}/V = 0$. Here $\langle - \rangle_{H_\Lambda^\omega}$ is the quantum Gibbs expectations with Hamiltonians H_Λ^ω.

• **Remark 2.2** If a many-body interaction satisfies the "local" gauge invariance:

$$[U_\Lambda, N_l(\phi_j)] = 0 \ ,$$

then U_l is a **function** of the **occupation number operators** $\{N_l(\phi_j)\}_{j\geq 1}$. For this reason it is called a "*diagonal interaction*".

• **Corollary 2.3** The *random* localised generalised (type ?) boson condensation occurs **if and only if** there is a generalised (type III) condensation in the extended (*kinetic-energy*) eigenstates. This a key to save the Bogoliubov theory in non-translation invariant, but a homogeneous random external potential.

2.3 The Second Main Theorem

• Let for any $A \subset \mathbb{R}_+$ the particle occupation measures m_Λ and $\tilde{m}_\Lambda$ are defined by:

$$m_\Lambda(A) := \frac{1}{V} \sum_{j:E_i \in A} \langle N_\Lambda(\phi_i^\omega)\rangle_{H_\Lambda^\omega} \ , \ \tilde{m}_\Lambda(A) := \frac{1}{V} \sum_{k:\varepsilon_k \in A} \langle N_\Lambda(\psi_k)\rangle_{H_\Lambda^\omega} \ .$$

• **Theorem 2.4**[Jaeck-Pulé-Zagrebnov]

$$m(dE) = \begin{cases} (\overline{\rho} - \rho_c)\delta_0(dE) + (e^{\beta E} - 1)^{-1}\mathcal{N}(dE) & \text{if } \overline{\rho} \geqslant \rho_c \ , \\ (e^{\beta(E-\mu_\infty)} - 1)^{-1}\mathcal{N}(dE) & \text{if } \overline{\rho} < \rho_c \ , \end{cases}$$

$$\tilde{m}(d\varepsilon) = \begin{cases} (\overline{\rho} - \rho_c)\delta_0(d\varepsilon) + F(\varepsilon)d\varepsilon & \text{if } \overline{\rho} \geqslant \rho_c, \\ F(\varepsilon)d\varepsilon & \text{if } \overline{\rho} < \rho_c. \end{cases}$$

with **explicitly** defined density $F(\varepsilon)$.

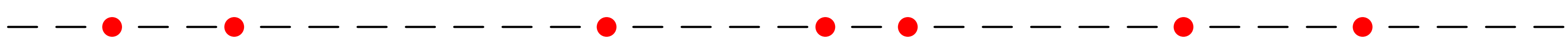
2.4 Example: BEC in One-Dimensional Random Potential. Poisson Point-Impurities

- For $d = 1$ *Poisson point-impurities*, $a > 0$:

$$v^{\omega}(x) := \int_{\mathbb{R}^1} \mu_{\tau}^{\omega}(dy) a\, \delta(x-y) = \sum_j a\, \delta(x - y_j^{\omega})$$

Proposition 2.5 Let $a = +\infty$. Then $\sigma(h^{\omega})$ is a.s. nonrandom, dense *pure-point* spectrum $\overline{\sigma_{p.p.}(h^{\omega})} = [0, +\infty)$, with IDS

$$\mathcal{N}(E) = \tau \frac{e^{-\pi\tau/\sqrt{2E}}}{1 - e^{-\pi\tau/\sqrt{2E}}} \sim \tau e^{-\pi\tau/\sqrt{2E}},\ E \downarrow 0, \text{(Lifshitz tail).}$$



- **Spectrum:**

$$(a.s.) - \sigma(h^\omega) = \bigcup_j \left\{\pi^2 s^2 / 2(L_j^\omega)^2\right\}_{s=1}^\infty$$

- Intervals $L_j^\omega = y_j^\omega - y_{j-1}^\omega$ are *i.i.d.r.v.* :

$$dP_{\tau, j_1, \ldots, j_k}(L_{j_1}, \ldots, L_{j_k}) = \tau^k \prod_{s=1}^k e^{-\tau L_{j_s}} dL_{j_s}$$

- **Eigenfunctions:**

One-particle **localized** quantum states $\{\phi_j\}_{j \geq 1}$, a basis in $L^2(\Lambda)$.

III.Generalized Bogoliubov c-numbers approximation

3.1 Existence of the approximating pressure

• Since randomness implies fragmented (or generalized type III) condensation, following the Bogoliubov approximation philosophy, we want to replace all creation/annihilation operators in the momentum states ψ_k with kinetic energy **less** than some $\delta > 0$ by c-numbers. Let $I_\delta \subset \Lambda_l^*$ be the set of all *replaceable* modes

$$I_\delta := \left\{k \in \Lambda_l^* : k^2/2 \leqslant \delta\right\},$$

and we denote $n_\delta := \sharp\{k : k \in I_\delta\}$. The number of quantum states n_δ is of the order V_l, since by definition of the Integrated Density of States: $n_\delta = V_l \ \nu_l^0(\delta)$.

• Let $\mathcal{H}_l^\delta$ be the subspace of $\mathcal{H}_l$ spanned by the set of ψ_k^l with $k \in I_\delta$, and P_δ be orthogonal projector onto this subspace. Hence, we have a natural decomposition of the total space $\mathcal{H}_l$ and the corresponding representation for the associated symmetrised Fock space:

$$\mathcal{H}_l = \mathcal{H}_l^\delta \oplus \mathcal{H}_l^\perp \ , \qquad \mathcal{F}_l \approx \mathcal{F}_l^\delta \otimes \mathcal{F}_l^\perp .$$

• Then we proceed with the Bogoliubov substitution $a_k \to c_k$ and $a_k^* \to \bar{c}_k$ for all $k \in I_\delta$, which provides an *approximating* (for the initial) Hamiltonian, that we denote by $H_l^{low}(\mu, \{c_k\})$.

• The partition function and the corresponding pressure for this approximating Hamiltonian have the form:

$\Xi_l^{low}(\mu, \{c_k\}) = \mathrm{Tr}_{\mathcal{F}_l^\perp} e^{-\beta H_l^{low}(\mu,\{c_k\})}$, $p_{l,\delta}^{low}(\mu, \{c_k\}) = \frac{1}{V_l} \ln \Xi_l^{low}(\mu, \{c_k\})$

.

• **Theorem 3.1 [Jaeck-Zagrebnov]** The c-numbers substitution for all operators in the energy-band I_δ does not affect the original pressure in the following sense:

$$\begin{aligned} \text{a.s.}-\lim_{l\to\infty} p_l(\beta,\mu) &= \lim_{\delta\downarrow 0}\liminf_{l\to\infty}\max_{\{c_k\}} p_{l,\delta}^{low}(\mu,\{c_k\}) \\ &= \lim_{\delta\downarrow 0}\limsup_{l\to\infty}\max_{\{c_k\}} p_{l,\delta}^{low}(\mu,\{c_k\}) \ . \end{aligned}$$

Note that the number of the substituted modes is of order V, since we let $\delta\downarrow 0$ after the thermodynamic limit.

• **Remark 3.2** The one-mode case: [Ginibre,Buffet-Pulé, Lieb-Seiringer-Yngvason].

THANK YOU FOR YOUR ATTENTION !