The Feigel Process : Lorentz-invariance, regularization, and experimental feasibility
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I. EXECUTIVE SUMMARY

The Feigel process refers to a recent theoretical analysis of vacuum fluctuations in a magneto-electrically active (ME) medium [1, 2], hereafter referred to as the "Feigel work". In such matter light can propagate without being absorbed, but its dispersion law is influenced by both an externally applied electric and magnetic field. We will define this later in a more precise way, and we restrict here by the remark that in a ME medium two optical effects can occur, among which one is believed to be crucial for the Feigel work. In particular, in ME media a difference exists between photons with wave vector $\mathbf{k}$ and $-\mathbf{k}$, but contrary to the Faraday effect - which needs a magnetic field only - this difference is independent of polarization. Contrary to optical activity, which requires a broken mirror symmetry of the matter, a fortunate property of the ME effect is that it can occur in *all* media subject to an electric and magnetic field, and even occurs in the quantum vacuum as we will see.

The presence of both an electric field $\mathbf{E}_0$ and a magnetic field $\mathbf{B}_0$ defines a third vector $(1/4\pi)\mathbf{E}_0 \times \mathbf{B}_0 \equiv \mathbf{S}_0$. Both under time-reversal $\mathcal{T}$, mirror reflection $\mathcal{P}$, as well as under charge conjugation $\mathcal{C}$ this vector behaves like a momentum density. The final result of the Feigel publication [1] is an expression for the momentum density $\rho\mathbf{v}$ (mass density ρ, velocity $\mathbf{v}$) of a homogeneous, infinite ME medium, in which only electromagnetic vacuum fluctuations reside,

$$\boxed{\rho\mathbf{v} \;=\; \frac{(\mu^{-1}+\varepsilon)}{32\pi^3}\,\frac{\hbar\omega_c^4\chi}{c_0^4}\,\mathbf{E}_0\times\mathbf{B}_0} \tag{1}$$

The optical constants μ and ε will be introduced later, χ is a ME coupling parameter, and ω_c is a frequency introduced in the Feigel work beyond which no ME effects are believed to occur. Feigel's conclusion is that the matter achieves a finite momentum density from the infinite vacuum sea, determined in direction and magnitude by the applied external fields. The press enthusiastically referred to this result as "momentum from nothing".

Many critical notes are in order before this surprising and important result can be appreciated and applied, and the work reported here summarizes the critical study we have undertaken. The evident first problem is the apparent Lorentz-*variance* of the final result (1). Both vectors $\rho\mathbf{v}$ and $\mathbf{S}_0$ as well as the frequency cut-off ω_c transform under a Lorentz transformation, and any physical law should in principle be invariant under this transformation. This is not trivially true for Eq. (1), an annoying complication especially because it was derived *appealing to* Lorentz-invariance in the first place.

The Feigel effect is thus controversial to say the least. Does it exist and how big is it? A complication is the Feigel theory actually diverges when calculating the momentum density. This divergence originates from Lorentz-invariance of the quantum vacuum which *imposes* an energy density proportional to ω^3 [2], whose integral diverges brutally as ω^4. To circumvent this problem an *ad-hoc* frequency cut-off ω_c was introduced by Feigel. The diverging energy density of electromagnetic vacuum fluctuations is well known, especially in the context of the Casimir effect [3]. This effect refers to the attracting force between two metallic plates in vacuum. It was predicted first by Casimir that a nonzero attractive force exists because the vacuum energy per unit surface is affected by the boundary conditions imposed by the metallic plates, and thus dependent on the distance L between the plates. Although the energy density itself is clearly infinite, its variation with L is actually finite, and can be calculated using the MacLaurin formula [2, 4] that estimates the difference between a Riemann integral and an infinite summation. Advanced regularization techniques, such as dimensional regularization and Riemann zeta function regularization, have recently been applied to regularize the energy density between the metallic plates [5]. It remains to be seen how these advanced methods affect the momentum density in the Feigel process, and how they affect the discussion of Lorentz-invariance. We will show that they provide a formula different from Eq. (1), and free from a cut-off. Not only is this mathematically more elegant, the regularization also provides a precise prediction for the order of magnitude, and is insensitive to unknown factors.

Finally, the Feigel work is obscured by many technical errors in both his theory and his derivations. His use of the Lagrange method to find momentum and pseudo momentum of electromagnetic fields and matter has already been criticized elsewhere [6]. A significant number of intermediate steps is inaccurate. For instance, angular integrals are incorrectly carried out. His intermediate result for momentum density $\rho\mathbf{v}+(4\pi)^{-1}\mathbf{E}\times\mathbf{H}$ (with $\mathbf{E}$ the electric field and $\mathbf{H}$ the macroscopic magnetic field) is subtly wrong. We will show in this report that the vacuum expectation value for the Poynting vector $(4\pi)^{-1}\mathbf{E}\times\mathbf{H}$ actually vanishes, and which would make $\rho\mathbf{v}$ thus also vanish. The additional incorrect use of the fluctuation-dissipation theorem for the vacuum fluctuations explains perhaps why this error was not noticed by Feigel.

A part of our criticism has already appeared in literature [7]. We have mainly criticized his claim that his Lagrange method "settles" the one-century old "Abraham-Minkowski" controversy[8] about what exactly is the momentum and angular momentum of a photon in matter. Maxwell's macroscopic equations allow several versions of momentum conservation, and different authors have proposed different arguments (symmetry of stress-tensor, Planck relation between momentum density $\mathbf{G}$ and Poynting vector $\mathbf{S}$) to make their choice. We believe that Feigel's work does not solve in any respect this controversy. On the other hand, we think that the study of the Feigel process is independent on this controversy.

A second criticism [7] is that stationary, homogeneous fields $\mathbf{E}_0$, $\mathbf{B}_0$ do - for an almost trivial reason - not impose the relation (1). In his published Reply [9] Feigel speculated that time-dependent fields $\mathbf{E}_0(t)$, $\mathbf{B}_0(t)$ might accelerate the matter. This option is not at all covered by his Lagrangian approach, since Lorentz-symmetry becomes very hard to capture in time-dependent, inhomogeneous media, but his remark is worth considering. This is the reason why in the following we shall always allow time-dependent, and possibly spatially varying external fields, and finally propose an experiment in this context. The fundamental test of Lorentz invariance, however, can only be carried out for homogeneous and stationary media.

The roadmap of this report is as follows.

The first chapter deals with the theoretical description of all optical phenomena. In section A we will establish a mathematically solid formulation of classical light propagation in bi-anisotropic media, and discuss the conservation of momentum when light interacts with bi-anisotropic matter. In section B we shall derive a Lorentz-invariant formulation for 4 magneto-electric optical effects, among which the Kerr effect, the Cotton-Mouton effect and the magneto-electric optical anisotropy. The latter is the crucial bi-anisotropic property that underlies the Feigel effect. In section C we shall address electromagnetical zero-point fluctuations in matter, and show how the fluctuation-dissipation formula establishes a direct link between the solutions of the classical Helmholtz-equation and the quantum-mechanical fluctuations of the electromagnetic field in vacuum. We need this to find the quantum-expectation value for the radiation momentum density. In section D we shall address the Feigel effect in the well-known Casimir-geometry. In this geometry we can come to regularized expressions, free from divergencies and unknown cut-offs. We shall conclude that the Feigel effect only exists in a squeezed vacuum, and is absent in the way Feigel originally proposed it. Unfortunately, the Feigel effect in a squeezed vacuum is too small to be observed with current techniques. Thus, finally in section E we propose a purely classical variant for the Feigel effect, for which observation seems within reach.

The second chapter outlines briefly the results of a profound literature study that we have undertaken to find magneto-electrically active materials, with crystal symmetries as close as possible to the one proposed originally by Feigel, and favorable to observe the Feigel effect. We propose an experiment to observe the Feigel effect, and give orders of magnitude for the different momenta of matter predicted by theory in our experiment. We finally conclude on the feasibility. The experiment is not part of this study, but will soon be carried out.

II. THEORETICAL STUDY

A. Constitutive equations and conservation laws

In this section we formulate conservation laws for bi-anisotropic media. The procedure is a standard generalization of classical electrodynamics of anisotropic media [10]. We have added this part to correct several inelegant inaccuracies in the Feigel work, to be able to start with a solid theory.

Bi-anisotropic media are described by a general linear "constitutive" relation between the macroscopic electromagnetic fields $\mathbf{D}, \mathbf{H}$, and the microscopic fields $\mathbf{E}, \mathbf{B}$,

$$\boxed{\begin{aligned}\mathbf{D} &= \varepsilon \cdot \mathbf{E} + \chi \cdot \mathbf{B} \\ \mathbf{H} &= -\chi^T \cdot \mathbf{E} + \mu^{-1} \cdot \mathbf{B}\end{aligned}} \tag{2}$$

The constitutive tensors ε and μ are assumed real-valued symmetric, the constitutive bi-anisotropic tensor χ is assumed real-valued. This excludes the presence of optical absorption. A case similar to the Feigel effect in the presence of atomic absorption and emission was recently discussed by us [11]. The Lorentz-invariance for spontaneous atomic emission has very recently been discussed [12]. In inhomogeneous media all constitutive tensors depend on the position vector $\mathbf{r}$. Time-dependence can be allowed as well provided the variation is much slower than the typical cycle oscillation of the electromagnetic fields, so that we can still work at constant frequency. The best-known case of optical bi-anisotropy is undoubtedly rotatory power, which can be described by the symmetric tensor $\chi_{ij} = g\delta_{ij}$, with g a pseudo scalar, induced by some microscopic chirality. In the Feigel work the anti-symmetric choice $\chi_{ij} = \chi(E_i^0 B_j^0 - B_i^0 E_j^0)$ was adopted, with χ a scalar. We will discuss Lorentz-invariance later. These relations are to be combined with four Maxwell's equations applied to harmonic fields $\exp(-i\omega t)$,

$$-i\omega \mathbf{B} = +\phi_{\mathbf{p}} \cdot \mathbf{E} \tag{3}$$

$$-i\omega \mathbf{D} = -\phi_{\mathbf{p}} \cdot \mathbf{H} - 4\pi \mathbf{J}_q \tag{4}$$

$$i\mathbf{p} \cdot \mathbf{B} = 0 \tag{5}$$

$$i\mathbf{p} \cdot \mathbf{D} = 4\pi \rho_q \tag{6}$$

where we have set $c_0 = 1$ and introduced the hermitian tensor operator $\phi_{nm,\mathbf{p}} \equiv i\epsilon_{nmk} p_k$ in terms of the fully anti-symmetric, third-rank Lévi-Civita tensor and the momentum operator $\mathbf{p} \equiv -i\nabla$; ρ_q is the macroscopic charge density and $\mathbf{J}_q$ is the macroscopic charge current density, that feature here as possible electromagnetic sources. It is easy to show the validity of the following Helmholtz equation,

$$\left[\omega^2 \varepsilon - i\omega \phi_{\mathbf{p}} \cdot \chi^T + i\omega \chi \cdot \phi_{\mathbf{p}} - \phi_{\mathbf{p}} \cdot \mu^{-1} \cdot \phi_{\mathbf{p}}\right] \cdot \mathbf{E} = -4\pi i\omega \mathbf{J}_q \,. \tag{7}$$

We can see that the terms proportional to χ - being odd in $\mathbf{p}$ - discriminate between wave vectors $\mathbf{k}$ and $-\mathbf{k}$. Since χ is typically a small parameter we shall treat these terms later as a first-order hermitian perturbation.

Conservation of energy will be first considered. The Maxwell equations (3) can be combined to give the well-known relation [10],

$$-4\pi \mathbf{J}_q \cdot \mathbf{E} = \nabla \cdot (\mathbf{E} \times \mathbf{H}) + \partial_t \mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \partial_t \mathbf{B}$$

If we now insert the constitutive equations with *time-independent* coefficients, and anticipate the relation $\mathbf{J}_q \cdot \mathbf{E} = \partial_t(\frac{1}{2}\rho v^2)$ for the rate of work done by the Lorentz force, we can re-arrange this into

$$\boxed{\partial_t \mathcal{E} + \nabla \cdot \mathbf{S} = 0} \tag{8}$$

where

$$\mathbf{S} = \frac{1}{4\pi} \mathbf{E} \times \mathbf{H} \tag{9}$$

is the Poynting vector, and

$$\mathcal{E} = \frac{1}{2}\rho v^2 + \frac{1}{8\pi}\mathbf{E}\cdot\varepsilon\cdot\mathbf{E} + \frac{1}{8\pi}\mathbf{B}\cdot\mu^{-1}\cdot\mathbf{B} \tag{10}$$

is the total energy density.

Most important for this work is the momentum balance. We will consider the general situation of inhomogeneous but non-absorbing media, possibly time-dependent, and get it here directly from the equations of motion rather than from the Lagrange formalism. The Maxwell equations (3) combine to

$$\partial_t(\mathbf{D}\times\mathbf{B}) = (\nabla\times\mathbf{H})\times\mathbf{B} - \mathbf{D}\times(\nabla\times\mathbf{E}) - \mathbf{J}_q\times\mathbf{B}$$

If we insert the constitutive equations, straightforward algebra leaves us with the conservation law,

$$\partial_t\mathbf{G} = \nabla\cdot\mathcal{T} - \mathbf{f}_q - \mathbf{f}_r \tag{11}$$

with $\mathbf{G} = (4\pi)^{-1}\mathbf{D}\times\mathbf{B}$ "some" momentum density of the radiation, $\mathbf{f}_q = \rho_q\mathbf{E} + \mathbf{J}_q\times\mathbf{B}$ the Lorentz force density that the electromagnetic field exerts on the macroscopic charges, the radiation force $\mathbf{f}_r$ whose n^{th} component equals

$$f_{r,n} = \frac{1}{8\pi}\left(\mathbf{E}\cdot\nabla_n\epsilon\cdot\mathbf{E} + \mathbf{B}\cdot\nabla_n\mu^{-1}\cdot\mathbf{B} - 2\mathbf{E}\cdot\nabla_n\chi\cdot\mathbf{B}\right) \tag{12}$$

created by any spatial inhomogeneity of the medium, and finally the stress-tensor,

$$\mathcal{T}_{ij} = \frac{1}{4\pi}(B_iH_j + D_iE_j) - \mathcal{E}\delta_{ij} \tag{13}$$

A second relation comes from Newton's second law applied to the matter, subject to the Lorentz force. If we write the macroscopic matter as a microscopic sum of moving particles with mass m_a and charge q_a it follows that

$$\begin{aligned}\partial_t\rho\mathbf{v}(\mathbf{r},t) &:= \partial_t\sum_a m_a\delta(\mathbf{r}-\mathbf{r}_a(t))\mathbf{v}_a(t)\\ &= -\nabla\cdot\rho\mathbf{v}\mathbf{v} + \sum_a q_a\delta(\mathbf{r}-\mathbf{r}_a)(\mathbf{E}_a + \mathbf{v}_a\times\mathbf{B}_a)\end{aligned}$$

If we *believe* in the existence of macroscopic fields $\mathbf{E}(\mathbf{r}), \mathbf{B}(\mathbf{r})$, whose variation is assumed slow over typical particle distances, we can replace the fields $\mathbf{E}_a$ and $\mathbf{B}_a$ at the particles by a slowly varying field,

$$\partial_t\rho\mathbf{v} + \nabla\cdot\rho\mathbf{v}\mathbf{v} = \rho_Q\mathbf{E} + \mathbf{J}_Q\times\mathbf{B}$$

where the index Q insists on the *total* charge, that is both microscopic *and* macroscopic (index q). We can now proceed by adopting the usual relations for macroscopic media and substitute

$$\rho_Q = \rho_q - \nabla\cdot\mathbf{P} \tag{14}$$
$$\mathbf{J}_Q = \mathbf{J}_q + \partial_t\mathbf{P} + \nabla\times\mathbf{M} \tag{15}$$

in terms of the microscopic polarization density $\mathbf{P} = (\mathbf{D}-\mathbf{E})/4\pi$ and the microscopic magnetization density $\mathbf{M} = (\mathbf{B}-\mathbf{H})/4\pi$. A long but straightforward calculation yields the conservation law for "*pseudo-momentum*",

$$\partial_t(\rho\mathbf{v} - \mathbf{P}\times\mathbf{B}) = -\nabla\cdot\mathcal{W} + \mathbf{f}_q + \mathbf{f}_r \tag{16}$$

with

$$\mathcal{W}_{ij} = \rho v_iv_j + (P_iE_j - B_iM_j) - \mathcal{E}\delta_{ij} + \frac{1}{8\pi}(\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B})\delta_{ij} \tag{17}$$

The pseudo-momentum $\int \mathrm{d}\mathbf{r}\,(\rho\mathbf{v} - \mathbf{P}\times\mathbf{B})$ is a conserved quantity *only* in the absence of sources (free charges) and in the absence of inhomogeneities. The rigourously conserved "momentum" can be found by adding the balance equations (11) and (16),

$$\boxed{\partial_t\left(\frac{1}{4\pi}\mathbf{E}\times\mathbf{B} + \rho\mathbf{v}\right) = \nabla\cdot(-\rho\mathbf{v}\mathbf{v} + \mathcal{T}_0)} \tag{18}$$

which contains the symmetric vacuum stress-tensor $\mathcal{T}_0$, with tensor elements

$$\mathcal{T}_{0,ij} = \frac{1}{4\pi}(E_iE_j + B_iB_j) - \frac{1}{8\pi}(\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B})\,\delta_{ij} \tag{19}$$

The "momentum" $\int d\mathbf{r}\ (\mathbf{E}\times\mathbf{B}/4\pi + \rho\mathbf{v})$ is conserved even in the presence of electromagnetic sources, in the presence of spatial inhomogeneities and smooth time-dependence of the constitutive parameters. We note that the Feigel work obtains $\int d\mathbf{r}\,(\mathbf{E}\times\mathbf{H}/4\pi + \rho\mathbf{v})$ for the total momentum (see Eq. 11 of the Feigel work), i.e. $\mathbf{H}$ in stead of $\mathbf{B}$. This is a highly unfortunate mistake since we will show later that the quantum expectation of the Poynting vector $\mathbf{S}$ vanishes. Secondly, the Feigel work is based on conservation of *pseudo-momentum*, which is not conserved in inhomogeneous media, neither in the presence of sources. These complications are unavoidable in (future) experimental realizations of the Feigel effect.

B. Lorentz-invariance of magneto-electric optics

One basic question is whether the Feigel process survives the sensitive test of Lorentz-invariance, which does not seem to be trivially satisfied from the prediction (1). The basic problem with Lorentz-invariance in the Feigel work is the lack of reference. The reference frame with respect to which $\mathbf{v}$ is supposed to be measured is not specified. Suppose that $\mathbf{E}_0$ and $\mathbf{B}_0$ are orthogonal (this is still a Lorentz-invariant statement). Then the momentum of the medium as predicted by Formula (1) points in the third direction. A Lorentz-boost with the appropriate *co-moving* velocity $\mathbf{v}$ along this direction yields $\mathbf{S}_0' = \mathbf{S}_0 - 2\mathcal{E}_0\mathbf{v}$ in the co-moving medium, with $\mathcal{E}_0$ the energy density associated with the externally applied electromagnetic field. We know that $-\rho\sqrt{1-v^2}$ is a Lorentz scalar so that ρ transforms only to order v^2. The righthand side of the Feigel formula transforms into $K\mathbf{S}_0((1-2\mathcal{E}_0K/\rho)$, whose relative change $\mathcal{E}_0K/S_0\rho$ can be seen to be absolutely negligible (see section II: of the order of 10^{-9}). Yet,the lefthand side changes by 100 %. This reveals the lack of a reference system in the Feigel work.

In this section we will use the term "Lorentz-invariance" - like Feigel - only up to orders v/c_0, which actually refers to Galilean invariance. The Lorentz transformation changes the electromagnetic field according to ($\gamma = 1/\sqrt{1-v^2}$),

$$\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{v}\times\mathbf{B}) - \frac{\gamma^2}{\gamma+1}\mathbf{v}(\mathbf{v}\cdot\mathbf{E}) \approx \mathbf{E} + \mathbf{v}\times\mathbf{B} \tag{20}$$

$$\mathbf{B}' = \gamma(\mathbf{B} - \mathbf{v}\times\mathbf{E}) - \frac{\gamma^2}{\gamma+1}\mathbf{v}(\mathbf{v}\cdot\mathbf{B}) \approx \mathbf{B} - \mathbf{v}\times\mathbf{E} \tag{21}$$

The last approximations are valid when $|\mathbf{v}| \ll 1$. Similar transformations can be proposed for the macroscopic fields $(\mathbf{D},\mathbf{H})$ [10]. If we insert these transformations into the constitutive equations, a little re-arranging shows that the three tensors must transform according to,

$$\begin{aligned}
\varepsilon' &= \varepsilon - (\epsilon\cdot\mathbf{v})\cdot\chi^T + \chi\cdot(\epsilon\cdot\mathbf{v}) \\
\chi' &= \chi + \mu^{-1}\cdot(\epsilon\cdot\mathbf{v}) - \varepsilon\cdot(\epsilon\cdot\mathbf{v}) \\
\mu'^{-1} &= \mu^{-1} - (\epsilon\cdot\mathbf{v})\cdot\chi + \chi^T\cdot(\epsilon\cdot\mathbf{v})
\end{aligned} \tag{22}$$

We introduced the anti-symmetric tensor $(\epsilon\cdot\mathbf{v})_{ij} = \epsilon_{ijk}v_k$, featuring again the Lévi-Civita tensor ϵ_{ijk} introduced earlier. A well-know consequence of this transformation is seen by putting $\mu = 1$ and by adopting a scalar dielectric constant ε. We see that χ' achieves an anti-symmetric term linear in the velocity. This leads to the so-called *Fizeau effect* that different directions of light propagation achieve different indices of refraction. Physically this is due to the Doppler effect.

The most elegant way to come to a Lorentz-invariant combination of the tensors ε, μ and χ is to built it in from the beginning. Two well-known Lorentz-scalars can be constructed, $\mathbf{E}^2 - \mathbf{B}^2$ and $\mathbf{E}\cdot\mathbf{B}$. The second is parity-odd and only even powers can be considered in the Lagrangian density. A Lorentz-covariant Lagrangian density can thus be proposed as,

$$\mathcal{L}(\mathbf{E},\mathbf{B},\mathbf{v}) = -\rho\sqrt{1-\mathbf{v}^2} + \frac{1}{2}\left(\varepsilon\mathbf{E}^2 - \mathbf{B}^2\right) - (\varepsilon-1)\mathbf{E}\cdot(\mathbf{v}\times\mathbf{B}) + \frac{\lambda}{2}\left(\mathbf{E}^2-\mathbf{B}^2\right)^2 + \frac{\nu}{2}(\mathbf{E}\cdot\mathbf{B})^2 \tag{23}$$

with λ,ν real-valued Lorentz-scalars, depending on the material. We do not claim that all ME materials must be described by this Lagrangian with appropriate scalar constants. We will simply show that this Lagrangian will provide a Lorentz-invariant description of the Feigel effect.

From the Lagrangian density (23) we can get the macroscopic fields $\mathbf{D},\mathbf{H}$ from the Euler-Lagrange formalism [13],

$$\mathbf{D} = \frac{\partial \mathcal{L}}{\partial \mathbf{E}} \quad , \quad \mathbf{H} = -\frac{\partial \mathcal{L}}{\partial \mathbf{B}}.$$

We have linearized the expressions so obtained into the genuine electromagnetic fields $\mathbf{E}, \mathbf{B}$, and the static, externally applied fields $\mathbf{E}_0, \mathbf{B}_0$,

$$\mathbf{E} \to \mathbf{E}_0 + \mathbf{E} \quad ; \quad \mathbf{B} \to \mathbf{B}_0 + \mathbf{B}$$

The end-result can be expressed as a bi-anisotropic medium (2) with tensors

$$\boxed{\begin{aligned} \varepsilon_{ij} &= \varepsilon\delta_{ij} + 2\lambda(\mathbf{E}_0^2 - \mathbf{B}_0^2)\delta_{ij} + 4\lambda E_{0,i}E_{0,j} + \nu B_{0,i}B_{0,j} \\ \mu_{ij}^{-1} &= \delta_{ij} + 2\lambda(\mathbf{E}_0^2 - \mathbf{B}_0^2)\delta_{ij} - 4\lambda B_{0,i}B_{0,j} - \nu E_{0,i}E_{0,j} \\ \chi_{ij} &= (\varepsilon - 1)\epsilon_{ijk}v_k - 4\lambda E_{0,i}B_{0,j} + \nu(\mathbf{E}_0 \cdot \mathbf{B}_0)\delta_{ij} + \nu B_{0,i}E_{0,j} \end{aligned}} \tag{24}$$

It is straightforward to show from the Lorentz transformation of *both* the external field and the relations (22) above that these constitutive relations are invariant under Lorentz transformation. For this we need the Cotton-Mouton effect (B_0^2 terms) and the Kerr effect (E_0^2 terms) in both the dielectric tensor and the magnetic permeability. The Heisenberg-Euler Lagrangian of the quantum vacuum is known to have $\nu = 7\lambda = e^4\hbar/45\pi m^4 c_0^7$ [14]. The Feigel work restricts itself to the case $\lambda = \nu/4$, and considers the (Lorentz-invariant) geometry $\mathbf{E}_0 \perp \mathbf{B}_0$, in which case the tensor χ is fully anti-symmetric and equal to $\chi_F = \epsilon \cdot [(\varepsilon - 1)\mathbf{v} + 2\nu\mathbf{S}_0]$. His choice for the constitutive equations is thus one Lorentz-invariant choice, among a much larger set, yet convenient for his purposes, since in homogeneous infinite media *symmetric* contributions to χ do not induce the Feigel effect anyway.

So what about the Lorentz invariance of the final Feigel formula? The relations (24) hold in an arbitrary reference frame with the external fields $\mathbf{E}_0$ and $\mathbf{B}_0$ expressed in the co-moving reference frame. Let us for simplicity adopt the geometry considered by Feigel: $\mathbf{E}_0 \perp \mathbf{B}_0$ and $\mathbf{v} \parallel \mathbf{E}_0 \times \mathbf{B}_0$. We choose $\mathbf{v}$ along the y-axis, $\mathbf{E}_0$ along the z-axis and $\mathbf{B}_0$ along the x-axis. A little algebra shows that the χ-tensor for a medium moving with velocity $\mathbf{v}$ takes the form [30]

$$\chi = (\varepsilon - 1)(\hat{\mathbf{z}}\hat{\mathbf{x}} - \hat{\mathbf{x}}\hat{\mathbf{z}}) + 4\pi\left(-4\lambda\hat{\mathbf{z}}\hat{\mathbf{x}} + \nu\hat{\mathbf{x}}\hat{\mathbf{z}}\right)(S_0 + 2v\mathcal{E}_0) \tag{25}$$

where now the external fields are measured in the reference frame, and $\mathcal{E}_0$ is the electromagnetic energy density associated with the external fields. An application of the Feigel work gives, *mutatis mutandis*, for the radiation momentum of the vacuum,

$$\langle 0|\frac{1}{4\pi}\mathbf{E} \times \mathbf{B}|0\rangle = \hbar K\left[(\lambda + 4\nu)\left(\mathbf{S}_0 + 2\mathbf{v}\mathcal{E}_0\right) + (\varepsilon - 1)\mathbf{v}\right]$$

with $K = \int d^3\mathbf{k}/(2\pi)^3\,\omega_{\mathbf{k}}$. This scalar is formally invariant under the Lorentz transformation $\mathbf{k} \to \mathbf{k} - \mathbf{v}t$, $\omega \to \omega - \mathbf{v}\cdot\mathbf{k}$, but actually diverges, which was already put forward as a problem of the Feigel work. The second problem was the obvious lack of reference in Eq. (1). We will now face both problems in the context of Lorentz-invariance. If we turn on - adiabatically - the external fields at $t = 0$ towards ststionary final values, with the medium at an initial velocity $\mathbf{v}(0)$ with respect to some reference frame, time-integration of Eq. (18) gives,

$$\rho\mathbf{v} - \rho\mathbf{v}(0) + \hbar K(\lambda + 4\nu)\left(\mathbf{S}_0 + 2\mathbf{v}\mathcal{E}_0\right) + \hbar K(\varepsilon - 1)[\mathbf{v} - \mathbf{v}(0)] = 0 \tag{26}$$

It is easily checked that this equation is Lorentz-invariant, provided some *Lorentz-invariant regularization* of the scalar K is adopted. In section D we will employ dimensional regularization to argue that $K = 0$ in an infinite medium. Hence $\mathbf{v} = \mathbf{v}(0)$, and the Feigel effect is not predicted to occur in an infinite medium. In section D we will investigate the Casimir geometry and show that in this case K varies inversely with the plate distance. We remark that the terms $2\mathbf{v}\mathcal{E}_0$ and $(\varepsilon - 1)[\mathbf{v} - \mathbf{v}(0)]$ do not appear in the Feigel work. The second is crucial to have Lorentz-invariance.

C. Fluctuation-dissipation theorem in bi-anisotropic media

To evaluate quantum expectation values for electromagnetic phenomena we need to express $\langle 0|E_i(\mathbf{r})E_j^*(\mathbf{r}')|0\rangle$, that is the quantum expectation value for the operator $E_iE_j^*$ in a zero photon field. This expression is supplied by the fluctuation dissipation theorem. Let G be the Green's tensor associated with the classical equation of motion (7) for the field $\mathbf{E}$,

$$G(\omega,\mathbf{p}) = \left[\omega^2\varepsilon - i\omega\phi_{\mathbf{p}}\cdot\chi^T + i\omega\chi\cdot\phi_{\mathbf{p}} - \phi_{\mathbf{p}}\cdot\mu^{-1}\cdot\phi_{\mathbf{p}}\right]^{-1} \tag{27}$$

The following fundamental relation is deeply related to the fluctuation-dissipation theorem [15],

$$\langle 0|E_i(\omega,\mathbf{r})E_j^*(\omega,\mathbf{r}')|0\rangle = -2\hbar\omega^2\langle\mathbf{r}|\mathrm{Im}\,G_{ij}(\omega,\mathbf{p})|\mathbf{r}'\rangle \tag{28}$$

We have replaced the real-valued electromagnetic fields by their complex-valued equivalents obtained by Hilbert transformation and applied the familiar counter-rotating wave approximation, that removes negative frequencies. The asterisk denotes complex conjugation. It is implicitly assumed here that vacuum fluctuations see the macroscopic medium, i.e. that the eigenmodes of the macroscopic Helmholtz equation are the ones that undergo second quantization. This is not at all an evident reality, and probably only true far away from microscopic absorption bands. It can be checked that for an infinite medium the expectation value for the electromagnetic energy density (10) is given by the familiar formula

$$\langle 0|\mathcal{E}|0\rangle = 2\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2}\hbar\omega_{\mathbf{k}} \tag{29}$$

which actually diverges, but which will be regularized to zero later. We next address the energy current density and assert the following theorem, worked out in Appendix A. In a homogeneous medium, with the constitutive parameters (24),

$$\boxed{\langle 0|\mathbf{S}|0\rangle = 0} \tag{30}$$

where $\mathbf{S} = \mathrm{Re}\,\mathbf{E}\times\mathbf{H}^*/(4\pi)$ is the Poynting vector (9), describing energy flow . In "normal" and homogeneous media this statement is trivially true, since vacuum fluctuations with wave vectors $+\mathbf{k}$ and $-\mathbf{k}$ are equally abundant. In bi-anisotropic, and/or inhomogeneous media this becomes less evident. During the short-term study we have not been able to prove the theorem in general (we even assert it to apply to inhomogeneous media), but we are convinced that a more general formulation exists. The theorem is physically reasonable since in the quantum vacuum no energy currents ought to flow.

We end this section with two remarks. First it is known that in genuine vacuum ($\varepsilon = 1$, $\mu = 1, \chi = 0, \rho = 0$) an *isotropic* vacuum radiation with spectral density ω^3 is imposed by Lorentz-invariance, as can be confirmed directly by the fluctuation-dissipation theorem. The generalization of this notion to anisotropic media is an interesting though difficult problem, largely relevant to the present project, but still unsolved to our knowledge.

Secondly, we insist that the Feigel work adopted the vacuum spectral density of a genuine vacuum, and did not apply the fluctuation-dissipation theorem adapted to his ME medium. This is a subtle error, because in view of his previous mistake, a *consistent* application of the theorem (30) would have led to a zero value for the "momentum acquired from nothing".

D. Regularized Feigel process: the Casimir geometry

Until now we have found Lorentz-invariant constitutive relations for ME effects, and concluded that the choice of Feigel is Lorentz-invariant, provided terms of order E_0^2 and B_0^2 are added to ε and μ. The next step is to address the UV catastrophe that occurs in the Feigel theory. This catastrophe is not new. Since Lorentz invariance imposes a spectral energy density that typically scales as ω^3, we get an infinite energy density when integrating over frequencies. Introducing a frequency-cut-off solves this problem, but this is 1.) theoretically inelegant, 2.) Lorentz-*variant* since the frequency cut-off is necessarily Lorentz-variant, and 3.) problematic when it comes to making quantitative predictions since the cut-off is not provided by theory but nevertheless comes in as a fourth power.

The Casimir effect is well-known to be insensitive to the UV catastrophe. This effect refers to the attractive force between two metallic plates, caused by the modified vacuum. The point is that the vacuum energy is infinite, but not the change as the distance between the plates is varied. This "fortunate cancellation" has encouraged theoreticians to find a rigorous regularization of the vacuum energy density [5], in the assumption that the UV divergency has no physical significance. In this section we will apply these methods to the Feigel effect, and come to a regularized prediction, free of unknown constants, and obeying Lorentz-invariance for any boost *along* the plates. The Casimir geometry is shown in Figure 1.

To evaluate the Feigel process in the Casimir geometry we consider Eq. (18) and note that it can be applied when the constitutive constants vary in either space or time. We can thus think of changing the distance between the metallic plates, or changing the fields $\mathbf{E}_0$, $\mathbf{B}_0$ slowly, with the ME slab initially at rest, and ask whether the latter starts moving, achieving a finite momentum per unit surface with respect to the metallic plates when all parameters become stationary again. Note that in this picture the plates play a triple role: they can be varied in distance and are as such a dynamical degree of freedom, they form a reference frame for the moving middle slab (rather fortunate for any discussion on Lorentz-invariance), and finally they provide a geometry in which divergencies have already been regularized successfully for the Casimir effect.

In the following we will use perturbation theory to find the frequency eigenvalues for the Casimir geometry (distance between the plates L) in the presence of a ME medium with thickness d, with the χ-terms in Eq. (7). We use the following result familiar from quantum-mechanics. If δK is a small perturbation of the hermitian operator K with real-valued eigenvalues ω_n^2 and with complete set of orthonormal eigenfuntions $|\mathbf{E}_n\rangle$, the change in eigenvalue is,

$$\delta\omega_n^2 = \langle \mathbf{E}_n| \cdot \delta K \cdot |\mathbf{E}_n\rangle \tag{31}$$

The eigenfunctions also change but will not be needed. The electromagnetic eigenmodes of the Casimir geometry in vacuum can be separated into TE and TM modes. The metallic boundary conditions impose that the tangential electric field vanishes on the plates ($z = 0, L$) as well as the normal magnetic field. The TE modes are,

$$\begin{aligned} \mathbf{E}_{n\mathbf{k}}(z,\mathbf{x}) &= \sqrt{\frac{2}{L}}\,\hat{\mathbf{k}}\times\hat{\mathbf{z}}\sin k_n z\exp(i\mathbf{k}\cdot\mathbf{x}) \\ \mathbf{B}_{n\mathbf{k}}(z,\mathbf{x}) &= \sqrt{\frac{2}{L}}\left(\frac{k}{\omega}\hat{\mathbf{z}}\sin k_n z + i\frac{k_n}{\omega}\hat{\mathbf{k}}\cos k_n z\right)\exp(i\mathbf{k}\cdot\mathbf{x}) \end{aligned} \tag{32}$$

and the TM modes

$$\begin{aligned} \mathbf{E}_{n\mathbf{k}}(z,\mathbf{x}) &= \sqrt{\frac{2}{L}}\left(\frac{k}{\omega}\hat{\mathbf{z}}\cos k_n z - i\frac{k_n}{\omega}\hat{\mathbf{k}}\sin k_n z\right)\exp(i\mathbf{k}\cdot\mathbf{x}) \\ \mathbf{B}_{n\mathbf{k}}(z,\mathbf{x}) &= -\sqrt{\frac{2}{L}}\,\hat{\mathbf{k}}\times\hat{\mathbf{z}}\cos k_n z\exp(i\mathbf{k}\cdot\mathbf{x}) \end{aligned} \tag{33}$$

with dispersion law $\omega = \omega_{n\mathbf{k}} \equiv \sqrt{k^2 + k_n^2}$ and $k_n = n\pi/L$: $n = 1, 2, \cdots$ for TE and $n = 0, 1, \cdots$ for TM. The perturbation operator is obtained from the Helmholtz equation (7),

$$\delta K = \omega(\epsilon\cdot\mathbf{p})\cdot\chi^T(z) - \omega\chi(z)\cdot(\epsilon\cdot\mathbf{p}) + [\varepsilon(z)-1]\omega^2 - (\epsilon\cdot\mathbf{p})\left[\mu(z)^{-1}-1\right](\epsilon\cdot\mathbf{p}) \tag{34}$$

with $\chi = -4\lambda\mathbf{E}_0\mathbf{B}_0 + \nu(\mathbf{E}_0\cdot\mathbf{B}_0)I + \nu\mathbf{B}_0\mathbf{E}_0 + (\varepsilon-1)(\epsilon\cdot\mathbf{v})$. We have ignored the contribution of $\mathbf{v}\mathcal{E}_0$ to χ as was mentioned in Eq. (25) which is necessary for Lorentz invariance but which is in reality very small. In this short research project we will assume that the ME slab possesses no normal polarizability but exhibits only ME effects. Hence $\varepsilon = 1$ and also the last "Fizeau" term disappears. The perturbation in eigenvalues can be straightforwardly obtained from Eq. (31). Only the terms proportional to χ are considered since the others, being even in p, can easily be shown not to generate a momentum current. We will simply give the result here,

$$\delta\omega_{n\mathbf{k}}^2(TE) = -4\omega k\left(4\lambda\mathbf{E}_0\cdot(\hat{\mathbf{k}}\times\hat{\mathbf{z}})(\mathbf{B}_0\cdot\hat{\mathbf{z}}) - \nu\mathbf{B}_0\cdot(\hat{\mathbf{k}}\times\hat{\mathbf{z}})(\mathbf{E}_0\cdot\hat{\mathbf{z}})\right)\times\frac{1}{L}\int_\chi dz\,\sin^2 k_n z \tag{35}$$

$$\delta\omega_{n\mathbf{k}}^2(TM) = -4\omega k\left(4\lambda\mathbf{B}_0\cdot(\hat{\mathbf{k}}\times\hat{\mathbf{z}})(\mathbf{E}_0\cdot\hat{\mathbf{z}}) - \nu\mathbf{E}_0\cdot(\hat{\mathbf{k}}\times\hat{\mathbf{z}})(\mathbf{B}_0\cdot\hat{\mathbf{z}})\right)\times\frac{1}{L}\int_\chi dz\,\cos^2 k_n z \tag{36}$$

where $\int_\chi dz$ is short for the integral $\int_{L/2-d/2}^{L/2+d/2} dz$ over the ME slab.

Equation (18) requires the evaluation of the zero-point expectation value for the "radiation" momentum density $\mathbf{E}^* \times \mathbf{B}$. The integral over z of $(\mathbf{E}^* \times \mathbf{B})/4\pi$ gives the momentum density of radiation per unit surface, which we shall denote by the vector $\mathbf{g}$. If we apply the Maxwell-relation (3) between $\mathbf{E}$ and $\mathbf{B}$ we see that $(\mathbf{E}^* \times \mathbf{B})(\omega)_i = -\epsilon_{ijk} E_j^*(\omega)\omega^{-1}(\epsilon \cdot \mathbf{p})_{kl} E_l(\omega)$. Since $\epsilon_{ijk}(\epsilon \cdot \mathbf{p})_{kl} = \delta_{il}p_j - \delta_{jl}p_i$ we get as an intermediate step,

$$\langle 0|(\mathbf{E}^* \times \mathbf{B})(\omega)_i|0\rangle = \omega^{-1} \sum_j \langle 0|\; E_j^*(\omega) p_i E_j(\omega) - E_j^*(\omega) p_j E_i(\omega)\; |0\rangle$$

and the application of the fluctuation-dissipation theorem (28) gives us,

$$\frac{1}{4\pi}\langle 0|(\mathbf{E}^* \times \mathbf{B})_i(\mathbf{r})|0\rangle = -\frac{1}{2\pi}\hbar \int_0^\infty \frac{d\omega}{2\pi}\,\omega \sum_j \left\{p_i \mathrm{Im}\, G_{jj}(\omega, \mathbf{r}) - p_j \mathrm{Im}\, G_{ij}(\omega, \mathbf{r})\right\} \tag{37}$$

where we recall that $p_j = -i\partial/\partial_j = \mathbf{k} - i\hat{\mathbf{z}}\partial_z$. We insert the spectral decomposition of the Green's tensor,

$$-\mathrm{Im}\, G_{ij}(\mathbf{x}, z, \mathbf{x}', z', \omega) = \sum_n \int \frac{d^2\mathbf{k}}{(2\pi)^2} E_i^{(n\mathbf{k})}(z) E_j^{*\,(n\mathbf{k})}(z)\; \pi\delta\left(\omega^2 - \omega_{n\mathbf{k}}^2\right)\, \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}$$

Since the eigenfunctions are normalized, the z -integral of the first term of Eq. (37) generates $\mathbf{k}\delta(\omega^2 - \omega_{n\mathbf{k}}^2)$ and can thus be conveniently expressed in terms of the eigenvalues only. This is unfortunately not the case for the second term, for which we still seem to need the perturbed eigenfunctions, which are somewhat harder to get. We can however apply the Maxwell relation $\nabla \cdot \mathbf{D} = 0$ to the first constitutive relation (2) with $\varepsilon = 1$, to find that $\mathbf{p}\cdot\ \mathbf{E} = -\mathbf{p}\cdot\chi\cdot\mathbf{B}$. Since this is already proportional to χ, we can use the unperturbed eigenfunctions and eigenvalues. We can re-arrange the formula for the radiative momentum density per unit surface to

$$\begin{aligned}\langle 0|\mathbf{g}|0\rangle \;&=\; \frac{1}{2}\hbar \int_0^\infty \frac{d\omega}{2\pi}\omega \sum_n \int \frac{d^2\mathbf{k}}{(2\pi)^2}\,\mathbf{k}\;\delta\left(\omega^2 - \omega_{n\mathbf{k}}^2\right) \\ &-\; \frac{\hbar}{8\pi i}\sum_n \int \frac{d^2\mathbf{k}}{(2\pi)^2} \int_0^L dz\, E_i^{(n\mathbf{k})}(z)\partial_j \chi_{jl} B_l^{*,\,(n\mathbf{k})}(z)\end{aligned} \tag{38}$$

We will treat the two terms separately and start with the first. We recall that the 2D vector $\mathbf{x}$ was defined parallel to the slab, and the z-coordinate normal to the plates. To simplify the analysis we choose the externally applied electric field $\mathbf{E}_0$ along the z-axis and the externally applied $\mathbf{B}_0$ along the x axis, such that $(4\pi)^{-1}\,\mathbf{E}_0 \times \mathbf{B}_0 \equiv \mathbf{S}_0$ points parallel to the plates (Fig 1).

First term of Eq. (38).

For the TE and the TM modes we can easily check from Eqs. (35) and (36) that

$$\omega_{n\mathbf{k}}^2(TE) \approx k_n^2 + \left(\mathbf{k} + 8\pi\omega\nu I_n^s\,\mathbf{S}_0\right)^2$$

$$\omega_{n\mathbf{k}}^2(TM) \approx k_n^2 + \left(\mathbf{k} + 32\pi\omega\lambda I_n^c\,\mathbf{S}_0\right)^2$$

where $I_n^s = L^{-1}\int_\chi dz \sin^2(k_n z)$ and $I_n^c = L^{-1}\int_\chi dz \cos^2(k_n z)$. Upon an appropriate change of base in the $\mathbf{k}$ integral we thus get

$$\mathbf{g}_1(TE) = -\hbar\nu\,\mathbf{S}_0 \sum_n \int \frac{d^2\mathbf{k}}{(2\pi)^2}\,\omega_{n\mathbf{k}} I_n^s \tag{39}$$

and for the TM modes

$$\mathbf{g}_1(TM) = -4\hbar\lambda\,\mathbf{S}_0 \sum_n \int \frac{d^2\mathbf{k}}{(2\pi)^2}\,\omega_{n\mathbf{k}} I_n^c \tag{40}$$

Since $I_n^{s,c} = (2L)^{-1} \int_\chi dz\,[1 \pm \sin(2\pi nz/L)]$ we see that the integrals diverge considerably in the UV. This divergence has been subject to at lot of study in literature [5]. We will apply here a number of regularization procedures that have recently been proposed by Kong and Ravndal [5]. The so-called dimensional regularization consists of pushing the relation

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d}\left(x^2+k^2\right)^{-p/2} \equiv I_x(d,p) = \frac{x^{d-p}}{(4\pi)^d}\frac{\Gamma(\frac{p-d}{2})}{\Gamma(\frac{p}{2})} \tag{41}$$

beyond its strict domain of validity $p > d$. The second regularization method, called zeta function regularization, applies to the discrete sum,

$$\sum_{n=0}^{\infty} n^{-s} = \zeta(s) \tag{42}$$

which is continued analytically to $s \le 1$ [16]. In particular, for the famous Casimir sum $\sum_{n=0}^{\infty} n^3 \to \zeta(-3) = 1/120$. The regularization of oscillation terms is less fancy. It is easily checked that for $\epsilon > 0$,

$$\sum_{n=0}^{\infty} \mathrm{e}^{-n\epsilon}\sin n\theta = \frac{1}{2}\,\frac{\sin\theta}{\cosh\epsilon - \cos\theta} \tag{43}$$

Hence for $\theta = 0, \pi$ this sum equals zero if we agree to take the limit $\epsilon \downarrow 0$ afterwards. For the Casimir effect this notion makes the frequency cut-off disappear [5]. If $\theta \neq 0, \pi$ the sum gives $\cos(\theta/2)/2\sin(\theta/2)$. Adapted to our needs, some algebra establishes the following regularization

$$\sum_{n=0}^{\infty} k_n^3 \sin^2 k_n z = \frac{\pi^3}{240L^3}\left(1 + \frac{15L}{\pi}\frac{\partial}{\partial z}\frac{\cos\pi z/L}{\sin^3\pi z/L}\right)$$

This expression can be seen to diverge as z^{-4} near the metallic plates.

With the proposed schemes we can regularize the expressions (39) and (40). We give here the final result,

$$\mathbf{g}_1(TE) = \frac{\pi^2}{1440}\frac{\hbar d\,\nu}{L^4}\left(1 - \frac{30L}{\pi d}\frac{\sin\pi d/2L}{\cos^3\pi d/2L}\right)\mathbf{S}_0 \tag{44}$$

and for the TM modes

$$\mathbf{g}_1(TM) = \frac{\pi^2}{360}\frac{\hbar d\,\lambda}{L^4}\left(1 + \frac{30L}{\pi d}\frac{\sin\pi d/2L}{\cos^3\pi d/2L}\right)\mathbf{S}_0 \tag{45}$$

This expression has the correct dimension of momentum per unit surface.

Second term of Eq. (38).

The unperturbed eigenfunctions have been given earlier. We have to recall that λ and ν are discontinuous on the slab boundaries. Using partial integration we can circumvent this problem. For the TM modes straightforward insertion gives for the momentum current density per unit surface along the y-axis,

$$\begin{aligned}\mathbf{g}_2(TM) &= \frac{4\hbar\lambda}{L}\mathbf{S}_0\int_\chi dz\sum_n\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{k_n^2}{\omega_{n\mathbf{k}}}\cos^2 k_n z\,(\hat{\mathbf{k}}\cdot\hat{\mathbf{y}})^2 \\ &= \frac{2\hbar\lambda}{L}\mathbf{S}_0\int_\chi dz\sum_n\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{k_n^2}{\omega_{n\mathbf{k}}}\cos^2 k_n z\end{aligned}$$

where we have averaged $\langle(\hat{\mathbf{k}}\cdot\hat{\mathbf{y}})^2\rangle = 1/2$ over angles. The dimensional regularization puts $I(2,1) = -k_n/2\pi$ according to Eq. (41). For the TE mode we get

$$\mathbf{g}_2(TE) = \frac{\hbar}{2L}\mathbf{S}_0\int_\chi dz\sum_n\int\frac{d^2\mathbf{k}}{(2\pi)^2}\left(\nu\frac{k^2}{\omega_{n\mathbf{k}}}\sin^2 k_n z + 4\lambda\frac{k_n^2}{\omega_{n\mathbf{k}}}\cos^2 k_n z\right)$$

Dimensional regularization gives $(2\pi)^{-2}\int d^2\mathbf{k}\, k^2\omega_{n\mathbf{k}} = k_n^3/6\pi$ and $(2\pi)^{-2}\int d^2\mathbf{k}\, k_n^2/\omega_{n\mathbf{k}} = -k_n^3/2\pi$. The integration over z is similar as above, and the regularized contribution of the second term to the momentum density per unit surface becomes,

$$\mathbf{g}_2(TM+TE) = \frac{\pi^2}{1440}\frac{\hbar d}{L^4}\left[(\nu-12\lambda)-(\nu+12\lambda)\frac{30L}{\pi d}\frac{\sin\pi d/2L}{\cos^3\pi d/2L}\right]\mathbf{S}_0 \tag{46}$$

The final result is by adding the contributions (44), (45), and (46),

$$\boxed{\mathbf{g} = \frac{\pi^2\hbar d}{720L^4}\left[(\nu-4\lambda)-(\nu+4\lambda)\frac{30L}{\pi d}\frac{\sin(\pi d/2L)}{\cos^3(\pi d/2L)}\right]\mathbf{S}_0} \tag{47}$$

for the radiation momentum of the zero-point fluctuations per unit surface. Equation (47) is the final mathematical result of this section. Somewhat surprisingly perhaps, we infer that also the *symmetric* part of χ - absent in homogeneous media - generates a small momentum per unit volume, and which is completely independent of the ME slab thicknesses d. Yet, for ME slab thicknesses $0 \leq d < L$ it can easily be checked that the expression is dominated by the *anti-symmetric* part of the magneto-electrical tensor χ, proportional to $\nu + 4\lambda$. It actually diverges as the plates start approaching the ME slab, a situation that we wish to ignore in the present context, but we note that if $d = L$ the diverging term regularizes to zero - in view of the discussion after Eq. (43) - so that only the symmetric part of χ remains. This is for instance true for a quantum vacuum between the Casimir plates, which is known to have a ME activity [13], with even a non-vanishing symmetric part of the χ-tensor.

Discussion

We shall now discuss the physical consequences of Eq. (47). It implies a finite radiation momentum per unit surface for a finite separation of the plates. The effect is Lorentz-invariant for any boost along the plates, provided the Lorentz transformation of all parameters is included (as discussed in a previous section). The transverse sizes L and d are not affected by a horizontal boost.

We recall the momentum balance equation (18) for radiation + matter, which was seen to be valid even for inhomogeneous media (and the middle plate is inhomogeneous in the z-direction), and even for time-dependent constitutive constants. Let us apply this notion to two Gedanken experiments.

First we imagine the plates to be infinitely separated ($L = \infty$), and the ME plate at rest with respect to these plates ($\mathbf{v}(0) = 0$), with the static or low frequency fields $\mathbf{E}_0$ and $\mathbf{B}_0$ switched on. We can now let the plates approach with opposite momentum until a finite distance L. (As a matter of fact the zero point fluctuations will do this for us by means of the Casimir effect, and we should actually prevent them from collapsing.) This converts vacuum energy into kinetic energy, but no momentum is put into the system, since the Casimir plates always have opposite momentum. Yet, the radiation momentum density $\mathbf{g}$ per unit surface changes, which must be compensated by the matter momentum density per unit surface $\rho d\mathbf{v}$. We conclude that the matter achieves the momentum density -$\mathbf{g}/d$ from the zero-point sea, with $\mathbf{g}$ given by (47). The Feigel effect thus occurs, but only at finite L: In our approach the diverging formula obtained by Feigel [1] for an infinite, homogeneous, and time-independent medium is completely regularized to zero. This can also be seen by ignoring the Casimir geometry, and by regularizing directly the expression obtained by Feigel (see also Eq. (26)),

$$\rho\mathbf{v} - \rho\mathbf{v}(0) = \frac{1}{4\pi}\hbar(\mu^{-1}+\varepsilon)\Delta\chi\mathbf{S}_0\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\omega_{\mathbf{k}} = I_0(3,-1) = 0$$

where the last equality follows from the dimensional regularization (41), and $\mathbf{v}(0)$ is the initial velocity with respect to the plates. The Feigel effect disappears but, quite satisfactorily, the dimensional regularization leaves us with a Lorentz invariant end-result.

We thus predict that the Feigel effect occurs only in a squeezed vacuum. An experimental verification of this statement would not only be an observation of the "Feigel effect" itself, it will also provide direct evidence for the regularization technique. In this context we emphasize that the observation of the Casimir effect [17] does not provide this support, since for the Casimir force - being the derivative of the diverging zero-point energy - the infinite constant formally drops out.

In a second experiment the plates are kept at a finite distance, but the fields $\mathbf{E}_0$ and $\mathbf{B}_0$ are switched off, with the medium at rest with respect to the plates. Upon turning on the fields slowly toward finite values $\mathbf{E}_0$ and $\mathbf{B}_0$, the total momentum per unit surface $\mathbf{g}(t) + \rho d\mathbf{v}(t)$ must again be constant in time, and we conclude for exactly the same "Feigel effect" as in the previous Gedanken experiment. An experimental verification of this statement would even more support the regularization procedure, since the regularized infinity is now actually time-dependent. The equivalence of the two Gedanken experiments avoids the occurrence of a hysteresis in the following cycle,

$$(\mathbf{S}_0 = 0, L = \infty) \rightarrow (\mathbf{S}_0 = 0, L) \rightarrow \underline{(\mathbf{S}_0, L)} \rightarrow (\mathbf{S}_0, L = \infty) \rightarrow (\mathbf{S}_0 = 0, L = \infty)$$

where only the middle underlined stage exhibits the Feigel effect.

E. A magneto-electrically active object in classical, diffuse light

Until now we have considered the Lorentz-invariance of the Feigel effect, and investigated the infinities that occur when considering it for zero-point fluctuations. Lorentz invariance has been an important physical test, but is not a stringent experimental condition. Controversial as it is, the Feigel effect in vacuum is mixed up with another controversial procedure, namely the (dimensional) regularization. In this last section we shall formulate the Feigel process in a "normal" scattering situation, in the presence of a monochromatic diffuse field. This picture is highly relevant for experiments, though more difficult to treat by Lorentz symmetry. One striking difference between the quantum vacuum and a classical, random, isotropic radiation field must be emphasized: If the object is a rest and exposed to an "isotropic" radiation originating from its far field, the object will see a dipolar radiation once it starts moving, by virtue of the Doppler effect. For the vacuum this is different: even moving objects see an isotropic vacuum, because the vacuum fluctuations are Lorentz-invariant! On the other hand, a strong similarity exists as well: We can show that field correlations obey a formula very similar to the fluctuation -dissipation formula (28). From that we can establish the "classical" equivalent of the Feigel effect without the need of a "controversial" regularization procedure.

An arbitrary but finite object is placed in a stationary "diffuse" field, which is random, but statistically isotropic and unpolarized, with energy density $\mathcal{E}(\omega)$ in the absence of the object. We shall prove the following theorem,

$$\boxed{\langle E_i(\mathbf{r},\omega) E_j^*(\mathbf{r}',\omega)\rangle = -\frac{2\pi\mathcal{E}(\omega)}{\omega} \mathrm{Im}\, G_{ij}(\mathbf{r},\mathbf{r}',\omega)} \tag{48}$$

with $G(\mathbf{r},\mathbf{r}',\omega)$ the Green's tensor of the Helmholtz equation (7), and $E_i(\mathbf{r},\omega)$ the component i of the electric field at position $\mathbf{r}$ at frequency ω. Note the similarity of this equation with the fluctuation-dissipation formula (28). The proof is given in Appendix B.

The relevance of a diffuse field in the context of the Feigel effect is obvious: without the object, the momentum of the radiation clearly vanishes. In the presence of the object, this might change if some magneto-electric activity is induced. It would be very interesting to apply (48) to for instance a Mie sphere [19] ("a water droplet"), that is slightly perturbed by ME effects. There is good hope that resonant effects of the sphere largely increase the Feigel effect. This is however largely beyond the scope of this project. We will consider a much more simpler situation which will allow us to conclude that the Feigel effect is induced. To this end we consider a finite object whose constitutive tensors ϵ, μ and χ differ only moderately from the (vacuum) environment. If G_0 is the vacuum Green's tensor of the Helmholtz equation (7) and $\delta K(\mathbf{r},\mathbf{p})$ the perturbation (34), we see that the Green's function in the presence of the perturbation is approximated by,

$$G(\omega, \delta K) = \left[G_0^{-1} - \delta K\right]^{-1} = G_0 + G_0 \cdot \delta K \cdot G_0 + \mathcal{O}(\delta K)^2$$

If we apply Eq. (37) for the radiation momentum density, it follows that,

$$\frac{1}{4\pi}(\mathbf{E}\times\mathbf{B}^*)(\mathbf{r},\omega)_i = -\frac{\mathcal{E}(\omega)}{2\omega^2}\langle \mathbf{r}|\ \left[p_i \mathrm{Im}\, G_{jj}(\omega) - p_j \mathrm{Im}\, G_{ij}(\omega)\right]\ |\mathbf{r}\rangle \tag{49}$$

and upon integrating over whole space, using the above expansion for G, we get for the total momentum density of the radiation,

$$\begin{aligned} g_i(\omega) &= -\frac{\mathcal{E}(\omega)}{2\omega^2}\int \frac{d^3\mathbf{k}}{(2\pi)^3} k_i \,\mathrm{Im}\,\mathrm{Tr}\, G_0(\mathbf{k})\cdot \chi_{\mathbf{k}\mathbf{k}'} \cdot G_0(\mathbf{k}) \\ &+ \frac{\mathcal{E}(\omega)}{2\omega^2}\int \frac{d^3\mathbf{k}}{(2\pi)^3} k_j \,\mathrm{Im}\, G_{0,in}(\mathbf{k})\cdot \chi_{\mathbf{k}\mathbf{k},nm} \cdot G_{0,mj}(\mathbf{k}) \end{aligned} \tag{50}$$

containing the ME matrix element,

$$\chi_{\mathbf{kk}} \equiv \langle \mathbf{k}|\chi(\mathbf{p},\mathbf{r})|\mathbf{k}\rangle = V\,B(k)\omega[(\epsilon\cdot\mathbf{k})\cdot\chi^T - \chi\cdot(\epsilon\cdot\mathbf{k})]$$

in terms of the volume V and the normalized structure factor $B(k)$ of the object (i.e. the normalized integral of $\exp(i\mathbf{k}\cdot\mathbf{r})$ over the object, thus $B(0)=1$). We emphasize that here the vector $\mathbf{g}$ stands for radiation momentum per unit volume, and not per unit surface as in the previous section. As for the Casimir geometry, we identify again two terms, the second being longitudinal (i.e. with $\mathbf{k}\cdot G \neq 0$) . We encounter no UV catastrophes. The vacuum Green's function [20] is $G_0(\omega,\mathbf{p}) = (I-\mathbf{pp})\left[(\omega+i0)^2 - p^2\right]^{-1}$. The first term gives rise to the momentum integral,

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,k^2\;\mathrm{Im}\,\frac{1}{\left[(\omega+i0)^2-p^2\right]^2} = \frac{3}{8\pi}\omega$$

and the second term requires,

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,,k^2\;\mathrm{Im}\,\frac{1}{(\omega+i0)^2-p^2} = -\frac{1}{4\pi}\omega^3$$

The final result reads,

$$\begin{aligned} g_i(\omega) &= -\frac{1}{48\pi}\mathcal{E}(\omega)\,V\,B(\omega)\epsilon_{ijk}\chi_{jk} \\ &= \frac{1}{48\pi}\mathcal{E}(\omega)B(\omega)\left\{4\pi(4\lambda+\nu)\mathbf{S}_0 + 2(1-\varepsilon)\frac{\mathbf{v}}{c_0}\right\}_i \end{aligned} \tag{51}$$

In the second equality we have inserted the constitutive equation (24) for χ. As in the previous section we apply the momentum conservation law (18) to conclude that

$$\rho\mathbf{v}(t) + \frac{1}{48\pi c_0}\mathcal{E}(\omega,t)B(\omega)\left\{(4\lambda+\nu)\,\mathbf{E}_0\times\mathbf{B}_0(t) + 2(1-\varepsilon)\frac{\mathbf{v}(t)}{c_0}\right\} = \text{constant} = 0 \tag{52}$$

We have re-introduced factors c_0 that had been put to unity for convenience, and put the constant equal to zero if we assume that $\mathbf{v}=0$ at the moment of switching on the field. Since the law (18) was seen to be valid in the presence of time-dependent sources and/or time-dependent constitutive parameters we were allowed to introduce slow time dependence in both the source and the fields $\mathbf{E}_0$ and $\mathbf{B}_0$. The constant in Eq. (52) is put to zero if on the moment of switching on either the source or the fields, the object is at rest. We conclude that the object starts moving along the vector $\mathbf{S}_0(t)$ as soon as both are switched on. The Feigel effect thus also exists for a ME object subject to an external field and placed in an isotropic radiation field. Note that here only the *asymmetric* part of the tensor χ comes in. We will estimate orders of magnitude later.

To obtain Eq. (52) we have considered the total radiation momentum accumulated around and in the object, but it is physically reasonable to assume that this momentum is localized around the object, and dragged along as the object starts moving. We have already seen that the momentum conservation Eq. (18) is valid for inhomogeneous external field, and thus for an inhomogeneous $\mathbf{S}_0(\mathbf{r},t)$. It is thus attractive to generalize Eq. (52) to inhomogeneous fields as follows,

$$\rho\mathbf{v}(\mathbf{R},t) + \frac{1}{48\pi c_0}\mathcal{E}(\omega,t)B(\omega)\left\{(4\lambda+\nu)\,\mathbf{E}_0\times\mathbf{B}_0(\mathbf{R},t) + 2(1-\varepsilon)\frac{\mathbf{v}(\mathbf{R},t)}{c_0}\right\} = \text{constant} = 0 \tag{53}$$

to be combined with $d\mathbf{R}/dt = \mathbf{v}$. This would imply that the object starts moving along the field lines of the vector $\mathbf{S}_0(\mathbf{R},t)$, with controllable speed. A fascinating idea with potential applications for optical tweezers, whose validity will be a topic of future research.

III. EXPERIMENTAL PERSPECTIVES

The major experimental challenge is to establish if the prediction by Feigel that a magneto-electric (ME) medium acquires linear momentum from the vacuum radiation field, is within reach of experimental verification. Feigel considers the case of an isotropic dielectric that is rendered magneto-electric by externally applied crossed static electric and magnetic fields. We propose to replace this medium by a ferrimagnetic, piezo-electric magneto-electric crystal that has identical forms for its constitutive tensors, but with much stronger effects. To detect the induced momentum, we propose to mount this crystal on an existing cantilever with a piezo-resistive readout (see Figure 2) subject to an alternating saturating magnetic field, and to detect phase sensitively the torque induced by the time-varying induced momentum. Phase-sensitive detection is a well-known method to measure small induced effects, and has been successfully applied by us in many other situations. Below we will determine what type of magneto-electric crystal is suitable, and what detection limit of the Feigel effect is achievable under realistic conditions.

Secondly, we wish to establish if the regularized prediction, derived by us above for the Casimir geometry, in Eq.(47), can be experimentally verified, by using the same material and setup as proposed for the verification of the original Feigel prediction, and by adding the plates.

Lastly, we wish to estimate the magnitude of the classical Feigel effect of ME diffusive scattering predicted by Eq. (52).

A. Induced magneto-electric effect in isotropic dielectrics

Using the Helmholtz equation Eq. (7) and the constitutive parameters Eq. (24), we can find out how different light polarizations propagate in different directions. If $\mathbf{E}_0$ is along the z-direction, $\mathbf{B}_0$ is along the x-direction we find for light propagating along the along the y-axis a linear birefringence

$$\Delta n \equiv n_x - n_z = \sqrt{\varepsilon\mu}\mu\left(\nu E_0^2 - 4\lambda B_0^2\right) + (\nu - 4\lambda)\,\mu E_0 B_0 + \frac{(\nu + 4\lambda\mu\varepsilon)\,B_0^2 - (4\lambda + \nu\mu\varepsilon)\,E_0^2}{2\varepsilon} \tag{54}$$

in which we can recognize a Cotton-Mouton birefringence

$$\Delta n_{\mathrm{CM}}\left(B_0^2\right) \equiv n_\parallel - n_\perp = n_x - n_z = \left(-4\sqrt{\varepsilon\mu}\mu\lambda + \frac{(\nu + 4\lambda\mu\varepsilon)}{2\varepsilon}\right) B_0^2 \tag{55}$$

a Kerr birefringence

$$\Delta n_K\left(E_0^2\right) \equiv n_\parallel - n_\perp = n_z - n_x = \left(\frac{4\lambda + \nu\mu\varepsilon}{2\varepsilon} - \sqrt{\varepsilon\mu}\mu\nu\right) E_0^2 \tag{56}$$

a magneto-electric linear birefringence

$$\Delta n_{\mathrm{MELB}}\left(E_0 B_0\right) \equiv n_B - n_E = n_x - n_z = (\nu - 4\lambda)\,\mu E_0 B_0 = (\chi_{12} + \chi_{21})\,\mu \tag{57}$$

We can also identify a magneto-electric anisotropy for unpolarized light

$$\Delta n_{\mathrm{MEA}}\left(E_0 B_0\right) \equiv n_{\mathbf{k}} - n_{-\mathbf{k}} = n_E - n_{-E} = (\nu + 4\lambda)\,\mu E_0 B_0 = (\chi_{12} - \chi_{21})\,\mu \tag{58}$$

If we assume that $\varepsilon \approx 1$ and $\mu \approx 1$ we can approximate $\Delta n_{CM} \approx (-2\lambda + \nu/2)\,B_0^2$, $\Delta n_K \approx (2\lambda - \nu/2)\,E_0^2$, $\Delta n_{MELB} \approx (\nu - 4\lambda)\,E_0 B_0$ and $\Delta n_{MEA} = (\nu + 4\lambda)\,E_0 B_0$. Clearly, all magneto-optic, electro-optic and magneto-electro-optic effects in rigid isotropic media are interrelated and we can deduce remarkably simple relations like:

$$\frac{\Delta n_{\mathrm{CM}}}{B_0^2} \approx -\frac{\Delta n_K}{E_0^2} \tag{59}$$

and

$$\frac{\Delta n_{\mathrm{MELB}}}{E_0 B_0} \approx 2\frac{\Delta n_{\mathrm{CM}}}{B_0^2} \tag{60}$$

B. Comparison with magneto-electric crystals

Having established the relation between the magneto-electric tensor of isotropic media in crossed fields and the observed optical phenomena, we can ask whether there exist crystals that by virtue of their intrinsic properties like magnetization and polarization, possess exactly the same magneto-electric tensor χ as derived in Eq. (24). In general one expects that the magnitude of the tensor elements for such crystals is many orders larger than for the case of externally applied fields to isotropic media. This expectation is based on the observation of a non-reciprocal magneto-electric birefringence of the order of 10^{-3} in non-centrosymmetric, anti-ferromagnetic Cr_2O_3 [21]. This crystal is the only one so far for which the real part of the magneto-electric tensor elements at optical frequencies has been determined. The low frequency values are also around 10^{-3}, which suggests that the magneto-electric effect is approximately frequency-independent, from DC up to optical frequencies. The strongest induced magneto-electric birefringence observed in liquids is 10^{-11}at $E = 10^5$ V/m and $B = 20$ T [22], i.e. 8 orders of magnitude smaller.

Among the 90 possible magnetic point groups, 32 point groups exist that contain a center of symmetry, which excludes the occurrence of magneto-electric effects [23]. This leaves us with 58 potential magnetic point groups that allow for a magneto-electric effect. The tensor form of the magneto-electric effect induced by external magnetic and electric fields in isotropic dielectrics,

$$\chi = \begin{pmatrix} 0 & a & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{61}$$

(where $a \propto E_0 B_0$ and $b \propto E_0 B_0$) is only found in crystals in the magnetic point groups $\underline{22}2$, $mm2$, $\underline{2m}m$ and $mm\underline{m}$, with $a \neq 0$ and $b \neq 0$ without external fields. Such crystals would therefore have exactly the same magneto-electric effects as rigid isotropic media in crossed electric and magnetic fields. The symmetric case $a = b$, i.e. $4\lambda = -\nu$ in the notation above, concerns crystals in the classes $4\underline{22}$, $4mm$, $\underline{\overline{4}}m2$, $4/\underline{m}mm$, $3\underline{2}$, $\underline{\overline{3}}m$, $3m$, $6\underline{22}$, $6mm$, $\underline{\overline{6}}m\underline{2}$ and $6/\underline{m}mm$. These would exhibit a higher tensor symmetry, and $\Delta n_{\text{MEA}} = 0$, so these crystal classes can not emulate dielectrics in crossed fields, where in general $\Delta n_{\text{MEA}} \neq 0$. More approximatively, all magnetic crystal classes where the matrix elements χ_{12} and χ_{21} are unequal and much larger than all other elements, will behave similarly to isotropic dielectrics in crossed external fields. The latter criterion eliminates the magnetic crystal classes $\underline{2}$, m, $\underline{2}/m$, 222, $\underline{mm}2$, $\underline{mmm}$, 422, $4\underline{mm}$, $\underline{\overline{4}2m}$, $4/\underline{mmm}$, 32, $3\underline{m}$, $\underline{\overline{3}m}$, 622, $6\underline{mm}$, $6/\underline{mmm}$, $\underline{42}2$, $\overline{4}2m$, $\underline{4/mm}m$, 23, $\underline{m}3$, 432, $\underline{\overline{4}3m}$ and $\underline{m}3\underline{m}$ where these elements are zero. This eliminates for instance the prototypical magneto-electric crystal Cr_2O_3 that belongs to the class $\underline{\overline{3}m}$. Note that the assignment of materials to a magnetic crystal class is a delicate issue. Not only will the magnetic symmetry depend on temperature, but the magnetization direction can depend on external influences, in particular on an external magnetic field that was applied, although in general an easy magnetization axis is dominant. Furthermore, effects of domains have to be considered, where each domain itself can be magneto-electric, but with adjacent domains generating opposite effects, so that the crystal as a whole does not show a magneto-electric effect. In general, such materials can be prepared in a single domain state by creating first a high symmetry state at high temperature, apply the electric/magnetic field and then slowly cool it down to the desired magneto-electric state. From the above, we see that only very few crystal classes facilitate an exact emulation of the behavior of isotropic dielectrics in crossed electric and magnetic fields. After a profound literature search, we propose $FeGaO_3$ as the most suitable candidate for such an emulation [24]. $FeGaO_3$ is an orthorhombic piezo-electric ferrimagnetic (below 280 K) crystal, with an electrical polarization along the b-axis and an easy magnetization direction along the c-axis. Identifying $c \to x$, $b \to y$ and $a \to z$, we deduce a magnetic point group $m\underline{2m}$ with the symmetry operations $\underline{2}_y$, $\overline{2}_x$ and $\underline{\overline{2}}_z$ and a magneto-electric tensor of the form given by Eq. (61) [25]. At low frequencies, a magneto-electric coefficient of $3 \cdot 10^{-4}$ has been observed [26]. More recently, the magneto-electric anisotropy of this material has been determined in absorption for radiation propagation along the a-axis at optical [27] and X-ray (7 keV, [28]) frequencies. Relative anisotropies in the absorption coefficient of the order of 10^{-3} have been observed at optical and X-ray resonances, implying

$$|\text{Im}\,(\chi_{12} - \chi_{21})| \approx 10^{-3} \text{Im}\, n \tag{62}$$

These observations impose through the Kramers-Kronig relations [10] anisotropies of the same order of magnitude in the real part $\text{Re}\,(\chi_{12} - \chi_{21})$. Quite complicated bi-signate line shapes were observed, so that no simple estimates for the magnitude of $\text{Re}\,(\chi_{12} - \chi_{21})$ could be obtained for frequencies outside the absorption regions. However, all these experimental results suggest that a value $\chi_{ij} \sim 10^{-4}$ can be expected over a wide frequency range from DC to X-rays. As an alternative, we propose the crystal $FeAlO_3$, which is isomorphous to $FeGaO_3$ and will therefore have the same magneto-electric tensor. Because the electronic transitions, generating both the magneto-electric effect and the absorption in $FeGaO_3$, are localized on the Fe^{3+}-ions, we expect very similar results for $FeAlO_3$.

C. Experimental feasibility of the Feigel effect

The relevant material parameters for $FeGaO_3$ at optical frequencies are :

$\rho = 4.5$ g/cm^3
$\mu \approx 1$
$\varepsilon \approx n^2 \approx 4$
$\Delta n_{MEA} \approx 10^{-4}$
$T_{FM} = 280\ K$

According to Eq. (1), the momentum density gained by the magneto-electric medium from the vacuum fluctuations is given by (in *cgs* units)

$$\rho v = \frac{1}{32\pi^3}\Delta n_{\text{MEA}}\frac{1+\varepsilon\mu}{\mu}\hbar k_c^4 = \frac{\pi}{2}\Delta n_{\text{MEA}}\frac{5\cdot 10^{-27}}{\lambda_c^4} = 78\ \text{g/cm}^2\text{s} \tag{63}$$

where a cut-off wavelength of $\lambda_c = 0.1$ nm was introduced, below which no more material response is supposed to exist. Note that magneto-electric activity was observed down to a wavelength $\lambda = 0,2$ nm [28] so that the cut-off λ_c cannot be much bigger than our choice, so that the above value for ρv is actually a lower limit. With the above value for the mass-density, we estimate a velocity of

$$v \approx 18\ \text{cm/s}. \tag{64}$$

A crystal of $FeGaO_3$ of mass $m = 10\ \mu g$ would therefore have a momentum

$$mv \approx 1.8\cdot 10^{-4}\ \text{gcm/s}. \tag{65}$$

We propose an experiment with a piezo-resistive-cantilever and phase-sensitive detection of the induced momentum by an AC magnetic field above the saturation field of $FeGaO_3$ (0.15 T along the c-axis) at a magnetic field frequency of $f = 60$ Hz. The predicted momentum change is equal to a periodic force

$$F_\omega = \frac{\partial mv}{\partial t} = 2\pi f mv \approx 0.07\ \text{gcm/s}^2 \tag{66}$$

on the cantilever. The beam length of the available cantilever is $l = 0.14$ mm, so the torque D on the cantilever would be

$$D_\omega = F_\omega \cdot l \approx 1\cdot 10^{-10}\ \text{Nm}. \tag{67}$$

The experimentally determined noise level of our cantilever readout system is about 500 nV/$\sqrt{\text{Hz}}$ between 1 Hz and 200 Hz at a sensitivity of about 10^7 V/Nm. This means that a signal-to-noise ratio of order unity can be obtained already after 20 ms integration. When integration extends to seconds, the predicted signal-to-noise ratio for the Feigel effect is large. The experiment is best done around 230 K, where Δn_{MEA} has its maximum value. Cooling of the environment to suppress contributions of the blackbody radiation field to the effect does not seem necessary. We recall that, according to the Feigel theory, the dominant contributions to the momentum transfer come from very high-energy photons, that are absent in blackbody radiation.

Upon observing a significant signal, an important check will be to change the direction of the b-axis of the crystal, which ought to change the sign of the signal. Another important check would be a measurement above T$_{FM}$= 280 K, where the magneto-electric coefficient of $FeGaO_3$ is known to vanish. Experimental verification or falsification of the Feigel effect by means of magneto-electric crystals seems therefore feasible. Note that the observation of this effect on isotropic dielectrics in crossed static fields with the same setup would require much longer integration times, of the order of days to weeks, which does not seem practical. A possible improvement of the setup would be a higher magnetic field frequency, which would require the construction of a special AC magnet.

D. Experimental feasibility of the Casimir-Feigel effect

In Eq. 47 a regularized Feigel effect has been obtained by us in the Casimir geometry. The regularization eliminates high energy photons that dominate in the Feigel theory, just like for the Casimir effect, which is generally accepted to be a low-frequency phenomenon. We predict for the momentum density of the magneto-electric slab with thickness d between two plates at a distance L,

$$\rho v = -g/d = \frac{h}{192L^4}\left(\frac{\chi_{12}+\chi_{21}}{30} + (\chi_{12}-\chi_{21})\,\frac{L}{\pi d}\frac{\sin(\pi d/2L)}{\cos^3(\pi d/2L)}\right) \tag{68}$$

Note that the momentum density diverges when $d \to L$. An optimistic, but not unrealistic situation would be $d \approx L/2$ which yields as estimate for $FeGaO_3$

$$\rho v \approx \frac{h}{192L^4}\Delta n_{\mathrm{MEA}} \approx \frac{2\cdot 10^{-32}}{L^4} \tag{69}$$

For $L = 10^{-3}$ cm, we finally obtain $\rho v = 4\cdot 10^{-20}$ g/cm^{2}s, as compared to the value $\rho v = 78$ g/cm$^2 s$ obtained by the Feigel theory, i.e. roughly 22 orders of magnitude smaller! Although a cantilever setup would in principle be suited to be implemented into the Casimir configuration, the calculated value for the momentum is too small to believe that experimental verification with our current cantilever setup is within reach. At finite temperatures the effect might be much bigger. A calculation of the Casimir-Feigel effect at finite temperatures is feasible, using the method employed for the calculation of the Casimir effect at finite temperatures [29].

E. Experimental feasibility of the classical Feigel effect

Eq. 52 predicts that a small object with magneto-electric properties in an isotropic monochromatic radiation field will be set into motion if the external fields are switched on. For this case, experimental verification is much easier, as high intensity monochromatic radiation fields can be easily generated and modulated, thereby facilitating experimental observation. Eq. 52 predicts for an object small compared to the wavelength ($B = 1$)

$$v = \frac{\chi_{12}-\chi_{21}}{24\pi\rho c_0^2 - (\varepsilon - 1)\,\mathcal{E}}\mathcal{E}c_0 \approx \frac{\Delta n_{\mathrm{MEA}}}{100\rho c_0} I \tag{70}$$

In the limit of very strong radiation fields($\mathcal{E} \gg 75\,\rho c_0^2$) this converges to $v = \Delta n_{\mathrm{MEA}} c_0$, independent of the radiation density. This corresponds to radiation fluxes of the order of $I \approx 10^{31}$W/m^2, which are not achievable in present experiments. For realistic values of the radiation intensity I the achieved momentum is just linear in I. For $I = \mathcal{E}c_0 = 10$ kW/cm^2, Eq. 70 predicts a velocity $v = 10^{-5}$ cm/s for an object made of $FeGaO_3$.

Experimental verification of the classical Feigel effect seems possible by the same cantilever technique as described above. A crystal of $FeGaO_3$ of mass $m = 10\ \mu g$ in an isotropic radiation field of 10 kW/cm^{2}would have a momentum

$$mv \approx 1\cdot 10^{-10}\ \mathrm{gcm/s}. \tag{71}$$

We propose this time to use a saturating DC magnetic field and periodically switch on and off the radiation field by means of a photoelastic modulator at a frequency $f = 50$ kHz. The predicted momentum change is equal to a periodic force

$$F_\omega = \frac{\partial mv}{\partial t} = 2\pi f mv \approx\ 3\cdot 10^{-5}\ \mathrm{gcm^2/s} \tag{72}$$

on the cantilever. The beam length of the available cantilever is $l = 0.14$ mm, so the torque D on the cantilever would be

$$D_\omega = F_\omega \cdot l \approx 5\cdot 10^{-12}\ \mathrm{Nm}. \tag{73}$$

Integration times of the order of seconds are sufficient to achieve a signal-to-noise ratio of unity. Important experimental checks are that the observed signal should change with the polarity of the (saturating) magnetic field and that the signal should vanish above T$_{FM}$= 280 K

IV. CONCLUDING REMARKS

This report is the result of a two months study on the Feigel effect, carried out in April/May 2005. We have investigated light propagation in bi-anisotropic media, and the conservation of momentum. A Lorentz-invariant description for the Feigel effect has been proposed. To eliminate the divergencies in the Feigel theory we have applied regularization techniques developed in quantum field theory. This has provided finite expressions for the Feigel effect in the well-known Casimir geometry consisting of two parallel metallic plates. We have also investigated a classical variant of the Feigel effect, which is experimentally more feasible. Finally, we have carried out a detailed literature study to find materials with favorable properties to observe or to reject the Feigel prediction.

Our conclusion is that the Feigel effect could in principle exist when external electric and magnetic field are slowly switched on. We do not confirm the final formula of the Feigel work, which expresses a dominant contribution of high-energy vacuum photons. We do confirm the Feigel effect in the Casimir geometry, but its order of magnitude is beyond experimental reach. We can consider the case treated by Feigel as the limit $L \longrightarrow \infty$ of the Casimir-Feigel geometry, treated by us in this report and conclude that Feigel's prediction that "momentum comes from nothing"' is too naive, and disregards recent developments of vacuum regularization. The most favorable situation that we could find where observation of a Feigel effect is possible is one in which a magneto-electric object is placed in an isotropic strong, classical radiation field.

APPENDIX A: ZERO ENERGY FLUX IN VACUUM

We proof here theorem (30).

Using the constitutive equations (2) and the Maxwell's equations, and by inserting a complete set of plane waves we can write for the i^{th} component of the Poynting vector,

$$S_i(\omega) = \frac{1}{4\pi}(\mathbf{E}\times\mathbf{H}^*)_i = \frac{1}{4\pi}\sum_{\mathbf{p}} \epsilon_{ijk}\left(\frac{\mu^{-1}\epsilon\cdot\mathbf{p}}{\omega} - \chi^T\right)_{kl} E_l(\omega,\mathbf{p})E_j^*(\omega,\mathbf{p})$$

whose vacuum expectation is, using (28),

$$\langle 0|S_i|0\rangle = -\frac{\hbar\omega}{2\pi}\sum_{\mathbf{p}} \epsilon_{ijk}\left(\mu^{-1}(\epsilon\cdot\mathbf{p}) - \omega\chi^T\right)_{kl}\ \operatorname{Im} G_{lj}(\omega,\mathbf{p})$$

The Green's tensor can be re-arranged to

$$G(\omega,\mathbf{p}) = \left[(\omega+i0)^2(\varepsilon - \chi\cdot\mu\cdot\chi^T) + (\mu^{-1}\cdot(\epsilon\cdot\mathbf{p}) - \omega\chi^T)^T\cdot\mu\cdot(\mu^{-1}\cdot(\epsilon\cdot\mathbf{p}) - \omega\chi^T)\right]^{-1}$$

The tensor $\mu\cdot\chi^T$ can always be written as the sum of a symmetric and an anti-symmetric tensor. The asymmetric part can always be written as the rotation over some axis $\mathbf{s}$, i.e. $(\mu\cdot\chi^T)^a = \epsilon\cdot\mathbf{s}$. The change of variable $\mathbf{p}\to\mathbf{p}+\mathbf{s}$ removes the asymmetric part, and we can thus assume $\mu\cdot\chi^T$ to be symmetric. For the Feigel work we have thus proved that $\langle 0|S_i|0\rangle = 0$, since his μ is scalar and his χ anti-symmetric, and the entire χ-tensor contribution can be eliminated by a change of variables.

It is at this moment that we restrict to the special case of our Lagrangian (23) with $\lambda \neq \nu/4$. The effect is evidently zero for $\chi = 0$ (the summation above over $\mathbf{p}$ is then odd in $\mathbf{p}$), so to second order of the applied fields $\mathbf{E}_0$, $\mathbf{B}_0$ we can neglect the ME terms in ϵ and μ^{-1}, and put $\varepsilon_{ij} = \varepsilon\delta_{ij}$. By first changing $\mathbf{p}\to-\mathbf{p}$ and then using the relation $A^T\cdot(1+A\cdot A^T)^{-1} = (1+A^T\cdot A)^{-1}\cdot A^T$, we get subsequently,

$$\begin{aligned} S_i \propto \operatorname{Tr}\epsilon_i\cdot A^T\cdot\operatorname{Im}G(A) &= \operatorname{Tr}\epsilon_i\cdot A^T\cdot\operatorname{Im}G(A^T) = \operatorname{Tr}\epsilon_i\cdot\operatorname{Im}G(A)\cdot A^T \\ &= \operatorname{Tr}A\cdot\operatorname{Im}G(A)\cdot\epsilon_i^T = -\operatorname{Tr}\epsilon_i\cdot A\cdot\operatorname{Im}G(A) \\ &= -S_i \end{aligned}$$

where we have inserted $A = ((\epsilon\cdot\mathbf{p})\omega - \omega\chi)$. We conclude that $S_i = 0$.

APPENDIX B: FIELD CORRELATIONS FROM ISOTROPIC RANDOM SOURCE

We proof here theorem (48).

We begin by considering a plane wave $s_{n\mathbf{k}}(\omega)g_n\exp(i\mathbf{k}\cdot\mathbf{x})$ with polarization g_n and complex amplitude $s_{n\mathbf{k}}$, incident on the object. In the presence of the object this is not an eigenfunction of the Helmholtz equation (7) at frequency $\omega = k$, but standard scattering theory [18] tells us that the appropriate eigenfunction is $E_{n\mathbf{k}}(r,\omega)$ is still related to the incident plane wave by means of the scattering operator. In the far-field of the object it converges to the incident plane wave with wave vector k and polarization g_n, plus a scattered spherical wave, but here we will need it mostly inside the object. The field correlation of the electric field between any two points in space is expressed as,

$$\begin{aligned} \langle E_i(\mathbf{r},\omega)E_j^*(\mathbf{r}',\omega)\rangle &= \int_{4\pi} d\Omega \int_{4\pi} d\Omega' \sum_{nn'}\left\langle s_n(\hat{\mathbf{k}},\omega)s_{n'}^*(\hat{\mathbf{k}}',\omega)\right\rangle E_{n\mathbf{k},i}(\mathbf{r},\omega)E_{n'\mathbf{k}',j}^*(\mathbf{r}',\omega) \\ &= \frac{1}{2}S(\omega)\int_{4\pi} d\Omega \sum_n E_{n\mathbf{k},i}(\mathbf{r},\omega)E_{n'\mathbf{k}',j}^*(\mathbf{r}',\omega) \end{aligned}$$

where we have used that $\left\langle s_n(\hat{\mathbf{k}},\omega)s_{n'}^*(\hat{\mathbf{k}}',\omega)\right\rangle = \frac{1}{2}\delta_{nn'}\delta(\Omega-\Omega')\,S(\omega)$ to describe an unpolarized isotropic random field. We see immediately from the equation above that in the absence of the object, we have the relation $4\pi S(\omega) = \langle|\mathbf{E}(\mathbf{r},\omega)|^2\rangle = \mathcal{E}_0$. We can rewrite this as,

$$\langle E_i(\mathbf{r},\omega)E_j^*(\mathbf{r}',\omega)\rangle = \frac{8\pi^2 S(\omega)}{\omega}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_n E_{n\mathbf{k},i}(\mathbf{r},\omega)E_{n'\mathbf{k}',j}^*(\mathbf{r}',\omega)\times\pi\delta\left(\omega^2-k^2\right)$$

Since $E_{n\mathbf{k}}(r,\omega)$ is the exact eigenfunction at eigenvalue k^2, we conclude that the righthand side is just the imaginary part of the (spectral decomposition of the) Green's function and,

$$\langle E_i(\mathbf{r},\omega)E_j^*(\mathbf{r}',\omega)\rangle = -\frac{8\pi^2 S(\omega)}{\omega}\,\mathrm{Im}\,G_{ij}(\mathbf{r},\mathbf{r}',\omega)$$

and the theorem is proved by inserting $S = \mathcal{E}/4\pi$.

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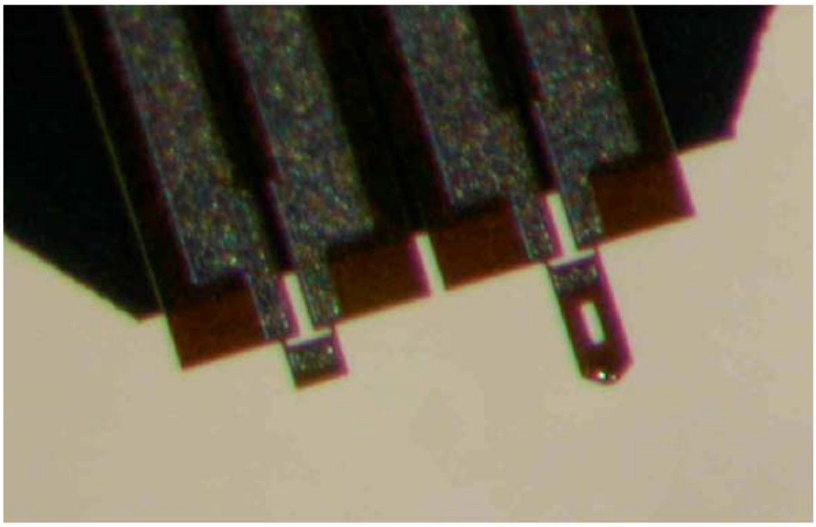
Figure 1

Figure 2



Picture of the differential piezo-electric cantilever used. On the left, the reference beam, with the piezo-resistive readout. On the right, the sample beam, with the piezo-resisitive readout. The two readout resistor sform one branch of a Wheatstone bridge, in this way compensating thermal drift. Width of the beams is 50 micrometer.