RECIPROCITY AND COHERENT BACKSCATTERING OF LIGHT

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The aim of this paper is to summarize, qualitatively and very globally, our present understanding of interference in multiple light scattering in disordered media, thereby hoping to bridge part of a gap between mathematical and physical approaches. A special emphasis is laid on the role of reciprocity in multiple light scattering, its relation to Coherent Backscattering in particular, and how to manipulate it with a magnetic field.

I. MAXWELL'S EQUATIONS

Let us start at the very beginning: the Maxwell equations for electromagnetic waves in macroscopic media ($c_0 = 1$) [1],

$$\begin{aligned} \partial_t \mathbf{D} &= \nabla \times \mathbf{H} - \mathbf{J} \ , \quad & \nabla \cdot \mathbf{D} &= \rho \ , \\ \partial_t \mathbf{B} &= -\nabla \times \mathbf{E} \ , \quad & \nabla \cdot \mathbf{B} &= 0 \ . \end{aligned} \tag{1}$$

Herein, ρ and $\mathbf{J}$ are the macroscopic charge density and charge current density. *Microscopic* charges and their movements have been put into the macroscopic fields $\mathbf{D}$ and $\mathbf{B}$. The relation of these fields to the original electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$ is given by the so-called constitutive equations,

$$\begin{aligned} \mathbf{D}(\mathbf{r},t) &= \boldsymbol{\varepsilon} \cdot \mathbf{E}(\mathbf{r},t) \\ \mathbf{H}(\mathbf{r},t) &= \boldsymbol{\mu}^{-1} \cdot \mathbf{B}(\mathbf{r},t) \\ \mathbf{J}(\mathbf{r},t) &= \boldsymbol{\sigma} \cdot \mathbf{E}(\mathbf{r},t) \ . \end{aligned} \tag{2}$$

The material parameters $\boldsymbol{\varepsilon}, \boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ - in general tensors of rank two - are called the dielectric constant, magnetic permeability and electrical conductivity. Eqs. (2) can be generalized to non-local situations and can given retardation effects. Scattering from the magnetic permeability hardly plays any role in day-life applications and we will set $\boldsymbol{\mu} = \mathbf{1}$. In what follows we will assume that $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are real-valued and local. The last property means that they are functions of the position (operator) $\mathbf{r}$ only. From this point mathematical and physical approaches deviate. In the mathematical approach (time reversal) symmetry is more obvious. The physical approach works better in applications and calculations since it requires less bookkeeping.

A. Maxwell à la Schrödinger: A Mathematician's Approach

Mathematicians like very much a first-order derivative with respect to time since it gives rise to a nice time-evolution problem familiar from Schrödinger operator theory. Maxwell's equations are of this type. A state vector can be introduced as,

$$\boldsymbol{\Psi}(\mathbf{r},t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \boldsymbol{\varepsilon}(\mathbf{r})^{1/2} \cdot \mathbf{E}(\mathbf{r},t) \\ \mathbf{B}(\mathbf{r},t) \end{pmatrix} \ . \tag{3}$$

Furthermore let,

$$\begin{aligned} \mathbf{K}(\mathbf{r},\mathbf{p}) = & \begin{pmatrix} 0 & \boldsymbol{\varepsilon}(\mathbf{r})^{-1/2} \cdot (\boldsymbol{\epsilon} \cdot \mathbf{p}) \\ (\boldsymbol{\epsilon} \cdot \mathbf{p}) \cdot \boldsymbol{\varepsilon}(\mathbf{r})^{-1/2} & 0 \end{pmatrix} \\ & -i \, \boldsymbol{\varepsilon}(\mathbf{r})^{-1/2} \cdot \boldsymbol{\sigma}(\mathbf{r}) \cdot \boldsymbol{\varepsilon}(\mathbf{r})^{-1/2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ , \end{aligned} \tag{4}$$

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in terms of the *operators* $\mathbf{r}$ and $\mathbf{p} = -i\nabla$ and the Lévi-Civita tensor of rank three $\boldsymbol{\epsilon}$. Maxwell's equations now take the form ,

$$i\partial_t \boldsymbol{\Psi}(\mathbf{r},t) = \mathbf{K}\cdot\boldsymbol{\Psi}(\mathbf{r},t) \ . \tag{5}$$

This equation looks like a Schrödinger equation in which the operator $\mathbf{K}$ plays the role of Hamiltonian or more exactly time-evolution generator ($\hbar = 1$). Given an initial wave function $\boldsymbol{\Psi}(\mathbf{r},0)$ the formal solution of Eq. (5) can be written as,

$$\boldsymbol{\Psi}(\mathbf{r},t) = \exp(-i\mathbf{K}t)\cdot\boldsymbol{\Psi}(\mathbf{r},0) \ . \tag{6}$$

This framework is very suited to discuss symmetry in Maxwell's equations, in particular because you can almost forget about light and have a look in books on quantum mechanics. If there is no finite conductivity and if the dielectric obeys $\boldsymbol{\varepsilon}(\mathbf{r}) = \boldsymbol{\varepsilon}^*(\mathbf{r})$, the operator $\mathbf{K}$ is an hermitean operator. Hence the operator $\exp(-i\mathbf{K}t)$ is unitary so that - like in Schrödinger theory - $|\boldsymbol{\Psi}(\mathbf{r},t)|^2$ is a conserved quantity. Translating back to the genuine electromagnetic fields this implies that

$$W(t) = \frac{1}{2}\int d\mathbf{r} \ \left[\mathbf{E}^*(\mathbf{r})\cdot\boldsymbol{\varepsilon}(\mathbf{r})\cdot\mathbf{E}(\mathbf{r}) + \mathbf{B}(\mathbf{r})^*\cdot\mathbf{B}(\mathbf{r})\right] \tag{7}$$

is conserved in time. This quantity is readily recognized as the total electromagnetic energy in the medium. In the mathematical theory of the Maxwell equations Eq. (7) is used to define a scalar product as in Schrödinger theory, and so to create an Hilbert space of square-integrable wave functions [2] [3].

A finite conductivity will destroy conservation of energy and gives rise to absorption. However, in the systems that we are interested in - dielectric substances like TiO_2 at optical frequencies - neglect of dissipation is a very good approximation. Conservation of energy is an extremely important notion for multiple scattering of light since it guarantees long-range diffusion (if no localization occurs by some other mechanism). The explicit occurrence of the dielectric tensor in the Eq. (7) has consequences for the propagation of light in the diffusive regime. We come back to this.

The framework discussed here is also suited to develop a light scattering theory [4] [5] [6]. One of the final goals of scattering theory is to find the scattering amplitude of a plane wave. The final result is - not not surprisingly - rather similar as in Schrödinger scattering theory, namely a Born series [2]. If we define the "potential" $\mathbf{V} \equiv \mathbf{K} - \mathbf{K}_0$ (here a 6×6 matrix), the Transition operator ("T-operator") at frequency ω is,

$$\mathbf{T}(\omega) = \mathbf{V} + \mathbf{V}\cdot\frac{1}{\omega + i0 - \mathbf{K}_0}\cdot\mathbf{V} + \cdots \ . \tag{8}$$

A plane wave with electric field polarization vector $\mathbf{g}$ and wave number $\mathbf{k}$ is here to be understood as a 6-column vector, which we shall indicate by a double bracket,

$$||\mathbf{g},\mathbf{k}\rangle\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}\mathbf{g}\\ \widehat{\mathbf{k}}\times\mathbf{g}\end{pmatrix}|\mathbf{k}\rangle \ . \tag{9}$$

The scattering amplitude ("T-matrix") for scattering from state $||\mathbf{g},\mathbf{k}\rangle\rangle$ into state $||\mathbf{g}',\mathbf{k}'\rangle\rangle$ is,

$$\langle\langle\mathbf{g},\mathbf{k}||\cdot\mathbf{T}(\omega)\cdot\left||\mathbf{g}',\mathbf{k}'\right\rangle\rangle \equiv t_{\mathbf{gkg'k'}}(\omega) \ . \tag{10}$$

In the next section we give an explicit expression of the scattering amplitude in terms of the scattered electric field only.

B. Maxwell à la Schrödinger: A Physicist's Approach

Generally speaking physicists prefer to have a closed equation for a directly measurable observable and - if possible - to eliminate all other variables. In case of electromagnetic waves most detectors measure the electric field amplitude $\mathbf{E}$. It is rather easy to combine the Maxwell equations in favor of one equation for the electric field,

$$\nabla\times\nabla\times\mathbf{E}(\mathbf{r},t) + \boldsymbol{\varepsilon}(\mathbf{r})\cdot\partial_t^2\mathbf{E}(\mathbf{r},t) = \boldsymbol{\sigma}(\mathbf{r})\cdot\partial_t\mathbf{E}(\mathbf{r},t) \ . \tag{11}$$

This is called the Helmholtz equation. Obviously a price has to be paid for eliminating the magnetic field $\mathbf{B}$: the wave equation has achieved a second-order time derivative, in sharp contrast to the approach given above.

One proceeds by inserting harmonic fields $\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r}) \exp(-i\omega t)$. Using the momentum operator $\mathbf{p} = -i\nabla$ and ignoring dissipation, the Helmholtz equation can be rewritten as,

$$\left\{\mathbf{p}^2\mathbf{\Delta_p} + [\mathbf{1} - \varepsilon(\mathbf{r})]\,\omega^2\right\} \cdot \mathbf{E}(\mathbf{r}) = \omega^2\mathbf{E}(\mathbf{r}) \ . \tag{12}$$

We introduced the transverse projection $\mathbf{\Delta_p}$ as $\mathbf{p}^2\mathbf{\Delta_p} \equiv \mathbf{p}^2 - \mathbf{pp}$. This equation looks very much like a normal Schrödinger eigenvalue equation provided one makes the identification ω^2 being the "light energy" and,

$$[\mathbf{1} - \varepsilon(\mathbf{r})]\,\omega^2 \equiv \mathbf{V}(\mathbf{r},\omega) \ , \tag{13}$$

the "light potential". A distinguished feature of this potential is that it depends on energy. It is well known in scattering theory that an interaction can depend on energy when some degree of freedom has been integrated out [7]. What remains is a so-called "optical potential". In our case it is the magnetic field that has been integrated out. The energy dependence can be shown to be mathematically equivalent to the explicit occurrence of the $\varepsilon(\mathbf{r})$ in the total energy (7) [8].

Having identified the link with normal Schrödinger scattering theory, it is tempting to write the transition operator of the electric field in the presence of a dielectric object as a Born series,

$$\mathbf{t}(\omega) = \mathbf{V}(\omega) + \mathbf{V}(\omega) \cdot \frac{1}{\omega^2 + i0 - \mathbf{p}^2\mathbf{\Delta_p}} \cdot \mathbf{V}(\omega) + \cdots . \tag{14}$$

Notice that $\mathbf{t}(\omega)$ is a 3×3-matrix, addressing the electric field only. The sandwich of this operator between two plane waves on the energy shell ($\mathbf{k}^2 = \omega^2$),

$$\langle \mathbf{g}, \mathbf{k} | \, \mathbf{t}(\omega) \, | \mathbf{g}', \mathbf{k}' \rangle \equiv t_{\mathbf{gkg}'\mathbf{k}'}(\omega) \ , \tag{15}$$

would then determine the scattering of an incident plane wave $|\mathbf{g}, \mathbf{k}\rangle$ to a plane wave $|\mathbf{g}', \mathbf{k}'\rangle$. A plane wave here refers to the electric field and is thus a three column vector; $\mathbf{g}$ is a polarization vector and can be complex-valued in case of circular polarization. It has always unit length.

Although Eq. (15) is merely based upon a physical analogy between the Schrödinger wave equation and the Maxwell equations, this final answer can be shown to coincide with the final result of rigorous light scattering theory [2], namely Eq. (10). This notion establishes a crucial link between mathematical and physical approaches. It demonstrates that we can use either one of them whenever necessary.

C. Time-Reversal Symmetry

Time-reversal symmetry is an important property of equations of motion. In Schrödinger theory a time-reversal operation is equal to complex-conjugation which is denoted by the anti-linear operator $\mathcal{C}$. By definition this operator satisfies,

$$\langle \mathbf{r} | \, \mathcal{C} \, | \Psi \rangle = \overline{\langle \mathbf{r} | \, \Psi \rangle} = \langle \Psi | \, \mathbf{r} \rangle \ , \text{ or } \ (\mathcal{C}\Psi)(\mathbf{r}) = \overline{\Psi(\mathbf{r})}$$

for a given wave function $\Psi(\mathbf{r})$. Because of the anti-linear property, Dirac notation must be used with care. Some well-known properties are: $\mathcal{C}^2 = 1$ and $\mathcal{C}\,|\mathbf{k}\rangle = |-\mathbf{k}\rangle$ for a plane wave $|\mathbf{k}\rangle$.

The formulation of time-reversal symmetry of light is most apparent in the mathematical formulation of Maxwell's equations. Inspection of Table 6.1 of Jackson [1] shows that under a time-reversal operation the magnetic field should also change sign. As a result, for light the time-reversal operator is,

$$\mathcal{T} = \begin{pmatrix} \mathcal{C}\mathbf{I} & 0 \\ 0 & -C\,\mathbf{I} \end{pmatrix} . \tag{16}$$

Time-reversal symmetry in light propagation can now be said to occur when,

$$\mathcal{T} \cdot \mathbf{K} \cdot \mathcal{T} = \mathbf{K} \ . \tag{17}$$

An isotropic finite conductivity (absorption) in $\mathbf{K}$ violates this identity since it gives rise to an imaginary part that changes sign under complex-conjugation. In that case,

$$\mathcal{T}\cdot\mathbf{K}\cdot\mathcal{T}=\mathbf{K}^{*} \tag{18}$$

A second interesting case occurs for the Faraday effect. This is an effect that will be considered as one of the main topics of this paper. It occurs when an *external* magnetic field (denoted by $\mathbf{B}_0$) is applied. This field is orders of magnitude larger (typically 20 T) than the magnetic field $\mathbf{B}$ of the light itself. However, it is independent of time and space and therefore disappears at most places in the Maxwell equations. The only place where it shows up is in the constitutive equation for $\mathbf{D}$. The dielectric tensor is given by [9],

$$\boldsymbol{\varepsilon}(\mathbf{r},\mathbf{B}_0)=\varepsilon(\mathbf{r})\ \mathbf{I}+g(\mathbf{r})\ i(\boldsymbol{\epsilon}\cdot\mathbf{B}_0)\ . \tag{19}$$

Herein $\varepsilon(\mathbf{r})$ is the normal dielectric constant and $g(\mathbf{r})$ the so-called gyromagnetic factor. If both quantities are real-valued this dielectric constant is hermitean and energy conservation is guaranteed. Physically this dielectric tensor gives rise to a rotation of the polarization vector of an electromagnetic plane wave. The direction of rotation is *dependent* of the direction of propagation (in sharp contrast to rotational power). In a homogeneous medium the rotation angle suffered by a linearly polarized plane wave with wave number k is VB_0L where $V=gk/2\sqrt{\varepsilon}$ is the Verdet constant and L the traversed length.

If we consider the field $\mathbf{B}_0$ as a given external parameter it is obvious that the imaginary term in Eq. (19) violates condition (17) for time-reversal symmetry. What we obtain is,

$$\mathcal{T}\cdot\mathbf{K}(\mathbf{B}_0)\cdot\mathcal{T}=\mathbf{K}(-\mathbf{B}_0)\ . \tag{20}$$

Hence time-reversal symmetry is broken in an external magnetic field.

One might criticize this conclusion by saying that $\mathbf{B}_0$ is also a magnetic field that also has to change sign in a time-reversal operation. In that case one arrives at the conclusion that time-reversal symmetry is *not* broken. We encounter here a situation for which we have time-reversal symmetry for the whole system (medium + magnet) but not for a subsystem (medium). For our purposes it is semantically convenient to restrict to the subsystem.

A number of properties of $\mathcal{T}$ can be verified for future use (we shall not proof them explicitly),

$$\mathcal{T}z\mathcal{T}=\overline{z} \tag{21}$$

$$\mathcal{T}\cdot||\sigma,\mathbf{k}\rangle\rangle=||\sigma,-\mathbf{k}\rangle\rangle \tag{22}$$

$$\langle\langle\sigma,\mathbf{k}||\cdot\mathcal{T}\cdot\mathbf{Q}\cdot\mathcal{T}\cdot||\sigma',\mathbf{k}'\rangle\rangle=\langle\langle\sigma',-\mathbf{k}'||\cdot\mathbf{Q}^{*}\cdot||\sigma,-\mathbf{k}\rangle\rangle \tag{23}$$

$$\mathcal{T}\cdot[z-\mathbf{K}]^{-1}\cdot\mathcal{T}=[\overline{z}-\mathcal{T}\cdot\mathbf{K}\cdot\mathcal{T}]^{-1} \tag{24}$$

Herein $||\sigma,\mathbf{k}\rangle\rangle$ is the Dirac notation for an six-dimensional electromagnetic plane wave with circular polarization $\sigma=\pm$, quite similar as in Eq. (9). Helicity is invariant under a time-reversal operation.

D. Reciprocity

Time-reversal symmetry is often confused with reciprocity. Reciprocity in light propagation can be heuristically defined as: "When you see me I can see you". For one case the difference is immediately clear: absorption. We demonstrated quite easily that absorption violates time-reversal symmetry. Nevertheless, if I can see you through an absorbing fog I have no doubt that you can see me as well. Hence reciprocity is still intact.

To express these ideas more quantitatively we recall that a signal emitted from a source traversing some medium and received by some detector is expressed by the T-matrix introduced in Eq. (15). Reciprocity for such a scattering set-up can be defined to occur when,

$$t_{\sigma\mathbf{k}\sigma'\mathbf{k}'}(\omega)=t_{\sigma'-\mathbf{k}'\sigma-\mathbf{k}}(\omega)\ , \tag{25}$$

for all possible values of σ and σ' and all directions of $\mathbf{k}$ and $\mathbf{k}'$. If detector and source are in the near field the scattering amplitude must also be considered "off-shell", where $\omega^2\neq\mathbf{k}^2$. In our definition reciprocity is obviously a property of the scattering amplitude whereas time-reversal symmetry is a property of the time-evolution. Sometimes reciprocity is defined in a more restricted sense for sources and detectors that are only sensitive to intensities and not the fields themselves (see e.g. the third chapter of Van de Hulst [10]). This definition comes rather close to the concept of "detailed balance" stating that in thermodynamic equilibrium any process has a reverse process with exactly the same rate. The relation of reciprocity to other properties in light propagation, like the Huygens' principle, has been discussed recently by de Hoop [11].

Using the results found in Eq. (21)-(24) we can investigate the validity of reciprocity. We shall allow both absorption and an externally applied magnetic field $\mathbf{B}_0$. To this end it is first convenient to establish that,

$$\mathcal{T} \cdot \mathbf{T}(\omega, \mathbf{B}_0) \cdot \mathcal{T} = \mathbf{T}^*(\omega, -\mathbf{B}_0) \ . \tag{26}$$

This follows straightforwardly from the definition (8) and Eq. (24). Using also that $\mathcal{T}^2 = 1$, Eq. (23) and the definition (10) it can be verified that,

$$t_{\sigma\mathbf{k}\sigma'\mathbf{k}'}(\omega, \mathbf{B}_0) = t_{\sigma'-\mathbf{k}'\sigma-\mathbf{k}}(\omega, -\mathbf{B}_0) \ . \tag{27}$$

We can draw the conclusion that *an external magnetic field breaks reciprocity in light propagation.* The physical consequences will be discussed later. It also follows from Eq. (27) that absorption does not destroy reciprocity, as conjectured earlier. In general it can be shown that time-reversal symmetry - as expressed by Eq. (17) - implies reciprocity, but not *vice versa.* Time-reversal symmetry in light scattering is thus stronger than reciprocity. Similarly, in equilibrium thermodynamics it is known that time-reversal symmetry of the microscopic Hamiltonian implies detailed balance [12] [13].

Sometimes it is convenient to define the 2×2 matrix $\mathbf{J}(\mathbf{k}, \mathbf{k}')$ with elements $J_{\sigma\sigma'}(\mathbf{k}, \mathbf{k}', \mathbf{B}_0) \equiv t_{\sigma\mathbf{k}\ \sigma'\mathbf{k}'}(\omega, \mathbf{B}_0)$. For this matrix the result (27) takes the form,

$$\mathbf{J}(\mathbf{k}, \mathbf{k}', \mathbf{B}_0) = \mathbf{J}^{\mathrm{t}}(-\mathbf{k}', -\mathbf{k}, -\mathbf{B}_0) \ . \tag{28}$$

A similar definition is possible with respect to a base with linear polarization. In that case the anti-transpose shows up, rather than the transpose as in Eq. (28). These results, with their application in multiple light scattering, have been discussed in detail by Martinez and Maynard [14].

II. MULTIPLE SCATTERING OF LIGHT

In this section we will discuss very briefly the physical method of calculating multiple light scattering in optically thick media. These media have the property of scattering incident light at least several times before leaving. We do not want to go in calculational details; Our aim is to set up a framework that can serve to elucidate the role of reciprocity in multiple light scattering.

We consider an inhomogeneous three-dimensional dielectric system from which we want to scatter light. The inhomogeneities can be thought of as small dielectric Mie spheres with some number density n. The question is how to describe multiple scattering from these objects if the solution of one such an object is known. Since we deal with disorder it is convenient to consider the average electric field $\langle \mathbf{E} \rangle$ and for the average intensity the set of (nine) Stokes correlations $\langle E_i \overline{E}_j \rangle$.

A. Average Amplitude

The propagation of the average electric field is determined by the average Green's function which, by Eq. (12), reads,

$$\begin{aligned} \langle \mathbf{G}(\omega) \rangle &= \left\langle \frac{1}{\varepsilon(\mathbf{r})\,\omega^2 + i0 - \mathbf{p}^2 \mathbf{\Delta}_\mathbf{p}} \right\rangle \\ &\equiv \frac{1}{\omega^2 + i0 - \mathbf{p}^2 \mathbf{\Delta}_\mathbf{p} - \mathbf{\Sigma}(\omega, \mathbf{p})} \ . \end{aligned} \tag{29}$$

We introduced the operator $\mathbf{\Sigma}(\omega, \mathbf{p})$ called the self-energy. Eq. (29) is called Dyson's equation. In an infinite medium, averaging restores translational symmetry and henceforth $\mathbf{\Sigma}(\omega, \mathbf{p})$ can only be dependent on frequency ω and momentum $\mathbf{p}$ (here still an operator). In the averaging process information is lost and consequently it is a irreversible operation. As a result, the operator $\mathbf{\Sigma}(\omega, \mathbf{p})$ (more precisely its matrix element in momentum space) must be complex-valued, the imaginary part being negative. The dielectric tensor of the effective medium can be defined as,

$$\varepsilon(\omega, \mathbf{p}) = \mathbf{I} - \frac{\mathbf{\Sigma}(\omega, \mathbf{p})}{\omega^2} \ . \tag{30}$$

In isotropic media we have $\boldsymbol{\varepsilon}(\omega, \mathbf{p}) = \varepsilon(\omega, \mathbf{p})\mathbf{I}$. Furthermore it is known that for most experimental applications one can neglect the momentum dependence in the dielectric function (called spatial dispersion; this approximation is as good as exact for a medium with point scatterers). The poles of the Dyson Green's function determine the coherent excitations in the light propagation. Due to the positive imaginary part of $\varepsilon(\omega, \mathbf{p})$ these excitations decay in space. It is customary to define the phase velocity v_{p} and the extinction mean free path ℓ_{e} according to,

$$\varepsilon(\omega)\,\omega^2 = \left(\frac{1}{v_{\mathrm{p}}} + \frac{i}{2\ell_{\mathrm{e}}}\right)^2 \ . \tag{31}$$

In real space the Green's function takes the asymptotic transverse form (in the far-field $kr \gg 1$),

$$\mathbf{G}(\mathbf{r}) = -\frac{\boldsymbol{\Delta}_{\mathbf{r}}}{4\pi r}\, \mathrm{e}^{i\omega r/v_{\mathrm{p}}}\, \mathrm{e}^{-r/2\ell_{\mathrm{e}}} \ . \tag{32}$$

Perturbation theory can be developed for the self-energy $\boldsymbol{\Sigma}(\omega, \mathbf{p})$ and it turns out that it can be represented by so-called irreducible diagrams [15] [16]. In lowest order of the particle density one obtains,

$$\boldsymbol{\Sigma}(\omega, \mathbf{p}) = n\mathbf{t}_{\mathbf{pp}}(\omega) \ . \tag{33}$$

Here $\mathbf{t}_{\mathbf{pp}}(\omega)$ is the scattering amplitude of one independent scatterer, defined in Eq. (15). In the low-density approximation one has ignored all recurrent scattering from two or more particles [17].

Physically the average amplitude describes the remnant of the incident beam due to extinction. By Eq. (31) the flux contained in the coherent beam decays exponentially with length scale ℓ_{e}. This extinction can be due to absorption. Since we have assumed to have no absorption it is entirely due to scattering out of the forward direction.

If only the coherent beam is detected one can pose the question whether or not this measurement obeys reciprocity. Obviously we then need to investigate whether the average scattering amplitude of the medium *as a whole*, denoted by $\langle T_{\sigma\mathbf{k}\sigma'\mathbf{k}'}(\omega)\rangle$, satisfies criterion (25). Using results obtained earlier we can quite smoothly conclude that such a coherent detection obeys reciprocity when no external magnetic field is present, simply because averaging will not change a statement that is true for every realization of the system. For a very large medium $\langle T_{\sigma\mathbf{k}\sigma'\mathbf{k}'}(\omega)\rangle$ is almost proportional to $\delta(\mathbf{k} - \mathbf{k}')$ so that we can only consider the forward-scattering channel with the coherent beam. Therefore a lot of information - contained in all scattering channels $\mathbf{k} \neq \mathbf{k}'$ - has been lost in the average field.

B. Diffuse Intensity

It is mathematically evident that the average of the square it not equal to the square of the average. Nevertheless in the physical literature the coherent beam, associated with the second, is not rarely mixed up with the diffuse intensity, denoted by the first. This happens because both of them are characterized by rather similar parameters. A thorough understanding of multiple light scattering requires not only the - rather obvious - mathematical difference, but also the physical difference. In this section we discuss the diffusive intensity, and compare it to the coherent beam.

The Green's function associated with the average diffuse, incoherent intensity is given by the four-rank tensor $\langle \mathbf{G}(\omega)\,\mathbf{G}^*(\omega')\rangle$. As was the case for the average amplitude the definition of a new object will help us. The Irreducible Vertex $\mathbf{U}(\omega, \omega')$ can be defined as,

$$\begin{aligned}\langle \mathbf{G}(\omega)\,\mathbf{G}^*(\omega')\rangle &= \langle \mathbf{G}(\omega)\rangle\langle \mathbf{G}^*(\omega')\rangle \\ &+ \langle \mathbf{G}(\omega)\rangle\langle \mathbf{G}^*(\omega')\rangle \cdot \mathbf{U}(\omega, \omega') \cdot \langle \mathbf{G}(\omega)\,\mathbf{G}^*(\omega')\rangle \ ,\end{aligned} \tag{34}$$

known as the Bethe-Salpeter equation. Like the self energy $\boldsymbol{\Sigma}$ defined earlier the four-rank tensor $\mathbf{U}$ is represented diagrammatically by irreducible diagrams, diagrams that cannot be cut into separate pieces [15]. The Bethe-Salpeter equation and the Dyson equation (29) can be combined to a transport equation in which $\mathbf{U}$ and $\boldsymbol{\Sigma}$ determine the collision operator. This method is equivalent to the more formal Liouville approach [18], but moreover distinguishes automatically between the average field and the average intensity, which are both measurable for light.

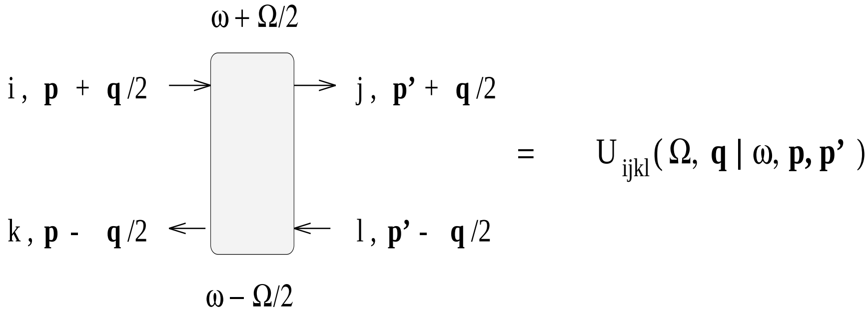

FIG. 1. Diagrammatic representation of the Irreducible Vertex $\mathbf{U}$ (denoted by the box). The convention of the polarization indices i, j, k, and l is indicated. Due to translational invariance only three independent momenta $\mathbf{p}$, $\mathbf{p}'$ and $\mathbf{q}$ remain. The latter determines the macroscopic deviation from homogeniety and is typically much smaller than the other two. The bottom line propagates in the reversed direction because it is the complex conjugate.

Multiple scattering is determined by the total $\mathbf{T}$-operator. If we recall the relation between $\mathbf{T}$-operator, the exact Green's function $\mathbf{G}(\omega)$ and the one in vacuum $\mathbf{G}_0(\omega)$,

$$\mathbf{G}(\omega) = \mathbf{G}_0(\omega) + \mathbf{G}_0(\omega) \cdot \mathbf{T}(\omega) \cdot \mathbf{G}_0(\omega) \ , \tag{35}$$

a little algebra establishes that,

$$\begin{aligned} &\mathbf{G}_0(\omega)\,\mathbf{G}_0^*(\omega') \cdot \{\langle \mathbf{T}(\omega)\,\mathbf{T}^*(\omega')\rangle - \langle \mathbf{T}(\omega)\rangle \langle \mathbf{T}^*(\omega')\rangle\} \cdot \mathbf{G}_0(\omega)\,\mathbf{G}_0^*(\omega') \\ &= \langle \mathbf{G}(\omega)\rangle \langle \mathbf{G}^*(\omega')\rangle \cdot \mathbf{R}(\omega,\omega') \cdot \langle \mathbf{G}(\omega)\rangle \langle \mathbf{G}^*(\omega')\rangle \ . \end{aligned} \tag{36}$$

In general we denote by $\mathbf{A}^*$ the hermitean conjugate of a matrix $\mathbf{A}$. We introduced the Reducible Vertex $\mathbf{R}(\omega,\omega')$ according to,

$$\mathbf{R}(\omega,\omega') = \mathbf{U}(\omega,\omega') + \mathbf{U}(\omega,\omega') \cdot \langle \mathbf{G}(\omega)\rangle \langle \mathbf{G}^*(\omega')\rangle \cdot \mathbf{R}(\omega,\omega') \ . \tag{37}$$

From Eq. (36) is follows that the scattered intensity is determined by the Reducible Vertex. The Dyson Green's functions on the right hand side of Eq. (36) represent physically the incoming and leaving waves near the surfaces of the sample that suffer from extinction.

Eq. (37) generates a geometric series in the Irreducible Vertex $\mathbf{U}$. One can say that $\mathbf{U}$ is the "building block" for multiple scattering. In general this Vertex is a complicated object, and it makes sense to investigate low-density expansions. It is convenient to first consider an infinite medium and to use the momentum representation. Due to translational symmetry, momentum is conserved and three independent momenta remain for the Vertex $\mathbf{U}$. In Fig. 1 we explain how the matrix element $\mathbf{U}(\Omega,\mathbf{q} \mid \omega,\mathbf{p},\mathbf{p}')$ is constructed from the operator $\mathbf{U}$. In lowest order of the density - the "Boltzmann approximation" - one gets,

$$\mathbf{U}^{\mathrm{B}}(\Omega,\mathbf{q} \mid \omega,\mathbf{p},\mathbf{p}') = n\mathbf{t}_{\mathbf{p}+\mathbf{p}'+}(\omega+\Omega/2)\,\mathbf{t}^*_{\mathbf{p}'-\mathbf{p}-}(\omega-\Omega/2) \ . \tag{38}$$

We abbreviated $\mathbf{p}^{\pm} = \mathbf{p} \pm \mathbf{q}/2$. With this approximation and the one in Eq. (33) the Bethe-Salpeter equation can be reduced to the Equation of Radiative Transfer.

We want to note that even the simplification of the Boltzmann approximation makes the solution of multiply scattered light from some geometry a numerical exercise. In the diffusion approximation one replaces the transport equation by only two coupled equations relating current density $\mathbf{J}_\omega(\Omega,\mathbf{q})$ and the electromagnetic energy density $\rho_\omega(\Omega,\mathbf{q})$. This approximation is valid for small Ω (much smaller than the optical frequency ω) and small $\mathbf{q}$ (much smaller than the optical wave number $\mathbf{k}$). The first is the equation of continuity (which in fact is always valid),

$$-i\Omega\rho_\omega(\Omega,\mathbf{q}) + i\mathbf{q}\cdot\mathbf{J}_\omega(\Omega,\mathbf{q}) = \mathrm{const.} \ . \tag{39}$$

The second one is a constitutive equation for the average current density [19],

$$\mathbf{J}_\omega(\Omega,\mathbf{q}) = -i\mathbf{D}(\omega)\cdot\mathbf{q}\,\rho_\omega(\Omega,\mathbf{q}) \ . \tag{40}$$

This equation yields the current as a gradient of the energy density. The diffusion tensor $\mathbf{D}$ is in general a tensor of rank two but is in normal cases simply scalar: $\mathbf{D} = D\mathbf{I}$. The diffusion constant of the light is given by the familiar relation,

$$D(\omega) = \frac{1}{3} v_{\mathrm{E}} \ell^* \ . \tag{41}$$

Herein is v_{E} the transport velocity and ℓ^* the transport mean free path, which are the only two remaining characteristic in the diffusion domain. Both are genuine properties of the average intensity and in general different from phase velocity and scattering mean free path in Eq. (15) associated with the average field. The Boltzmann expression for the transport mean free path is well-known,

$$\ell^* = \frac{\ell_{\mathrm{e}}}{1 - \langle \cos\theta \rangle} \ , \tag{42}$$

where $\langle \cos\theta \rangle$ is the average cosine of the scattering angle of one single particle. In particular scatterers with large forward-scattering have a transport mean free mean much larger than the extinction length. The Boltzmann expression for the transport velocity is less well-known. Several equivalent mathematical expressions can be given for it [8]. The physically most transparent one is,

$$v_{\mathrm{E}} \approx \frac{v_{\mathrm{p}}}{1 + \tau_\varphi v_{\mathrm{p}} / \ell_{\mathrm{e}}} \ , \tag{43}$$

in which $v_{\mathrm{p}}/\ell_{\mathrm{e}}$ is recognized as the inverse mean free collision time, and τ_φ is the Wigner phase-delay time of one scatterer. This time measures the delay in scattering [20] (later formulated mathematically by Jauch et al. [21]) and can be very large - even larger than the collision time - in resonant scattering [23]. Hence the transport velocity can be very slow.

The validity of the conservation law (39) is mathematically obvious because it holds true before averaging and must thus be true after averaging. Nevertheless, microscopically it is due to a rather elegant scattering identity, namely the fact that the t-matrix, defined in Eq. (15) satisfies the Optical Theorem. The Optical Theorem reads [4],

$$-\frac{\mathrm{Im}\, t_{\mathbf{gkgk}}(\omega)}{\omega} = \sum_{\mathbf{g}'} \int \mathrm{d}^2 \widehat{\mathbf{k}}' \ \frac{\left| t_{\mathbf{gkg}'\mathbf{k}'}(\omega) \right|^2}{(4\pi)^2} \ . \tag{44}$$

In multiple scattering, this relation establishes an important link between coherent beam and diffuse intensity: what is lost by the coherent beam reappears as diffusive light. In general, energy conservation gives rise to a highly non-trivial relation between self-energy and Irreducible Vertex [22], known as a Ward Identity.

Combination of constitutive equation and conservation law yields for the energy density a familiar diffusive expression,

$$\rho(\Omega, \mathbf{q}) \sim \frac{1}{-i\Omega + D(\omega)\, \mathbf{q}^2} \ . \tag{45}$$

In the diffusion approximation the Reducible Vertex $\mathbf{R}(\omega)$ is replaced by a similar hydrodynamic expression,

$$\begin{aligned} \mathbf{R}(\Omega, \mathbf{q} \mid \omega, \mathbf{p}, \mathbf{p}') &= \mathbf{U}^{\mathrm{B}}(\Omega, \mathbf{q} \mid \omega, \mathbf{p}, \mathbf{p}') \\ &+ \ \mathrm{const.}\ \frac{|\mathbf{d}\rangle \langle \mathbf{d}| \, .}{-i\Omega + D(\omega)\, \mathbf{q}^2} \end{aligned} \tag{46}$$

The single-scattering term is kept in Eq. (46) for further use, but obviously it is negligible with respect to the second in the diffusive regime. The eigenfunction $|\mathbf{d}\rangle$ with long range diffusion is - in isotropic media - proportional to δ_{ij} meaning that diffusive light is unpolarized [24]. All others (in fact eight) decay exponentially in space. Cases where it is necessary to go beyond the diffusion approximation in order to explain experimental results are rare.

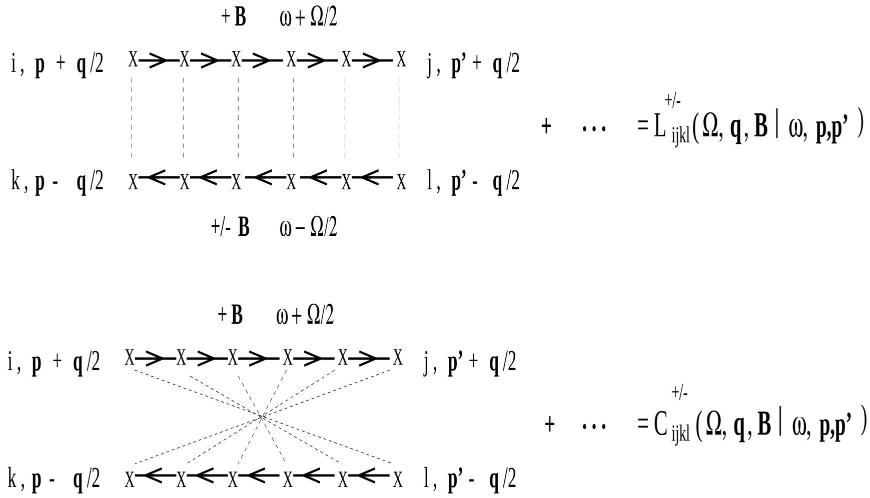

FIG. 2. Diagrammatic representation of the Ladder diagrams **L** and the most-crossed diagrams **C**. Only one diagram of the whole infinite geometric series is shown. The top line denotes the electric field, the bottom line its complex conjugate. Crosses denote the t-matrix of the Helmholtz equation. Dashed lines connect identical scatterers. Bold line denote the Dyson Green's tensor. We have allowed for two opposite directions of the magnetic field of the bottom line.

C. Reciprocity in Diffuse Scattering

The role of reciprocity in radiative transfer is well appreciated in literature. Despite its simple physical contents it gives rise to rather unexpected and sometimes non-trivial relations. For a good summary with references we refer to the third chapter of Van de Hulst [10]. For an elegant application in radiative transfer we can recommend a paper by Hovenier [25].

In the literature of radiative transfer reciprocity is defined with detectors and sources sensible for the Stokes parameters $\langle E_i E_j^* \rangle$. Reciprocity relations then relate various elements of the Müller matrix connecting the Stokes variables of incident and outgoing light. However, we have defined reciprocity earlier in Eq. (25) to be a property of the amplitude. In what follows we will first argue that *standard radiative transfer theory does not obey such reciprocity.* In restoring this property we discover a new phenomenon in multiple light scattering: *Coherent Backscattering.*

Let us consider the Boltzmann approximation (38). For simplicity we consider an infinite medium. This is by no means essential for the discussion, but it is convenient to take advantage of translational symmetry. The solution for the Reducible Vertex $\mathbf{R}(\Omega, \mathbf{q} \mid \omega, \mathbf{p}, \mathbf{p}')$ is shown diagrammatically in Fig. 2. Due to their outlook these diagrams are often called the *Ladder diagrams.* Using reciprocity (for the moment we assume that no magnetic field is applied externally) a topologically equal diagram can be constructed by reversing the direction of propagation of both the field (top line) and its complex-conjugate (bottom line). This provides us with the identity,

$$R_{ijkl}(\Omega, \mathbf{q} \mid \omega, \mathbf{p}, \mathbf{p}') = R_{jilk}(\Omega, -\mathbf{q} \mid \omega, -\mathbf{p}, -\mathbf{p}') \ . \tag{47}$$

In this identity source and detector are interchanged. The identity (47) is not only true in the Boltzmann approximation but must be valid in general since reciprocity applies no matter how difficult the light paths are that we consider. In the Boltzmann approximation they are necessarily self-avoiding, meaning that a light path never comes back to the same particle.

A second reciprocity relation that must be valid in general is obtained by reversing the direction of propagation of only the complex-conjugate wave $\mathbf{E}^*$,

$$R_{ijkl}(\Omega, \mathbf{q} \mid \omega, \mathbf{p}, \mathbf{p}') = R_{ijlk}\left(\Omega, \mathbf{p} + \mathbf{p}' \mid \omega, \frac{\mathbf{p} - \mathbf{p}' + \mathbf{q}}{2}, \frac{\mathbf{p}' - \mathbf{p} + \mathbf{q}}{2}\right) \ . \tag{48}$$

Let us verify whether this identity is satisfied by Boltzmann theory. Reversing the bottom line of any of the Ladder diagrams (except for the single scattering part $\mathbf{U}^{\mathrm{B}}$) makes us wind up with a topologically different diagram: it is irreducible and thus part of $\mathbf{U}$. However, this contribution has not been taken into account in the Boltzmann approximation (38). Hence *Boltzmann theory, and the equation of radiative transfer in particular, does not obey reciprocity.*

The set of diagrams that has been obtained here is called the set of *most-crossed diagrams* which we shall denote by $\mathbf{C}$. Physically it signifies an interference between two equal light paths but propagating in opposite directions. We can make an attempt to improve Boltzmann theory and to add these diagrams to the Irreducible Vertex. We have to beware not to double-count the single-scattering contribution $\mathbf{U}^{\mathrm{S}}$ since only for this event carrying out the procedure above gives us back - by the reciprocity relation (25) - the same single scattering. To this end let us write,

$$\mathbf{U}(\Omega, \mathbf{q} \mid \omega, \mathbf{p}, \mathbf{p}') \rightarrow \mathbf{U}^{\mathrm{B}}(\Omega, \mathbf{q} \mid \omega, \mathbf{p}, \mathbf{p}') + \mathbf{C}(\Omega, \mathbf{q} \mid \omega, \mathbf{p}, \mathbf{p}') \ , \tag{49}$$

and

$$\mathbf{R}(\Omega, \mathbf{q} \mid \omega, \mathbf{p}, \mathbf{p}') = \mathbf{U}(\Omega, \mathbf{q} \mid \omega, \mathbf{p}, \mathbf{p}') + \mathbf{L}(\Omega, \mathbf{q} \mid \omega, \mathbf{p}, \mathbf{p}') \ . \tag{50}$$

The reciprocity expressed by Eq. (48) now implies that,

$$\begin{aligned} &C_{ijkl}(\Omega, \mathbf{q} \mid \omega, \mathbf{p}, \mathbf{p}') \\ &\qquad = L_{ijlk}\left(\Omega, \mathbf{p}+\mathbf{p}' \mid \omega, \frac{\mathbf{p}-\mathbf{p}'+\mathbf{q}}{2}, \frac{\mathbf{p}'-\mathbf{p}+\mathbf{q}}{2}\right) \ . \end{aligned} \tag{51}$$

In the diffusion approximation (46) this equality can be expressed as,

$$\begin{aligned} L_{ijkl}(\Omega,\ \mathbf{q} \mid \omega, \mathbf{p}, \mathbf{p}') &\sim \text{const.}\ \frac{d_{ik} d_{lj}.}{-i\Omega + D\mathbf{q}^2} \ , \\ C_{ijkl}(\Omega,\ \mathbf{q} \mid \omega, \mathbf{p}, \mathbf{p}') &\sim \text{const.}\ \frac{d_{il} d_{kj}.}{-i\Omega + D(\mathbf{p}+\mathbf{p}')^2} \ . \end{aligned} \tag{52}$$

In particular, reciprocity implies that the diffusion constant D in both expressions is the same. Notice the far-reaching consequence of this observation: the final outcome of the transport equation, here a diffusion constant, features in the collision operator!

There is one disadvantage by simply adding the most-crossed diagrams on top of normal Boltzmann theory. It is well known that the Boltzmann theory obeys conservation of energy. To put it otherwise, from the Boltzmann transport equation one can proof the existence of an equation of continuity for the electromagnetic energy. Obviously this must also be true for the exact solution but this one is too difficult to be of any practical use. Adding the most-crossed diagrams to Boltzmann theory restores reciprocity but is doomed to destroy energy conservation. It turns out that finding an approximate transport theory satisfying both is not easy, since one has to generalize notions like Ladder diagrams and most-crossed diagrams in a self-consistent way. The present status is a self-consistent equation for the diffusion constant, mainly in the context of electron propagation [22] and acoustic waves [26]. The formulation of such a self-consistent transport equation forms an enormous challenge since it offers the possibility of a phase transition. That might enable us to establish the link between transport theory and Anderson localization of waves. [27].

D. Interference in Multiple Light Scattering

Now that reciprocity has led us to introduce a new object $\mathbf{C}$ on can pose the question how to deduce its existence experimentally. Simplest would be to observe it directly either in reflection or transmission; more difficult is to observe it indirectly, for example in a transport quantity like the diffusion constant. To the first we will refer as *Coherent Backscattering*, to the second as *Weak Localization.* Both names are frequently used through each other but we believe that a separation is in order [28]. The first has only been measured for light, the second sofar only for electrons. To explain the basic ideas of Weak Localization we spend a short subsection on it.

1. *Coherent Backscattering of Light*

From the reciprocity relation (51) one can deduce that the vertices $\mathbf{C}$ and $\mathbf{L}$ constitute - given an incident wave in direction $\mathbf{k}$ with circular polarization σ - *exactly equal* contributions of scattered light at backscattering $-\mathbf{k}$ in the same polarization channel σ. Physically this so-called Coherent Backscattering is due to the 100 % constructive interference of the two time-reversed waves. Physically it is obvious that this effect will dephase rapidly away from backscattering. Being an interference effect (characteristic length scale the wavelength $\lambda = 2\pi/k$) in multiple scattering (with relevant length scale the transport mean free path ℓ^*) a typical value for the dephasing angle is [29] [30],

$$\Delta\theta \approx \frac{1}{k\ell^*} \; . \tag{53}$$

The enhancement factor E in Coherent backscattering can be defined as the ratio of top over background. The background it given by the Vertices $\mathbf{L}$ and $\mathbf{U}^{\mathrm{B}}$ evaluated at backscattering; the enhancement by $\mathbf{C}$. Thus,

$$E_{xx'} = 1 + \frac{C_{xx'}\,(\mathrm{ref})}{L_{xx'}\,(\mathrm{ref}) + U^{\mathrm{B}}_{xx'}\,(\mathrm{ref})} \; . \tag{54}$$

Here x and x' denote incident and final polarization channels.

After initiating work by Kuga and Ishimaru [31], Coherent Backscattering of light has first been observed some 10 years ago [32] [33] [34] and after that an explosion in literature occurred. Coherent Backscattering of light has been measured time-resolved [35] [36], in a magnetic field [37] [38], with absorption [39], in two dimensions [40], in nematic liquid crystals [41] and on rough surfaces [42]. Dependence on polarization has been investigated experimentally [43]. It turned out that opposite polarization channels - either circular or linear - hardly exhibit any enhanced backscattering, and enhancements of only 112 % are reported. High orders of scattering give a very narrow cone and are therefore difficult to resolve experimentally. For this reason the measured enhancement factors did not always coincide with theoretical predictions based on Eq. (54). Only recently an experiment had sufficient resolution (± 1 milliradian) to resolve almost all orders of scattering and to address the numerical value of the enhancement factor in more detail [44].

Theoretically there is also a vast amount of information available in literature. The first predictions of the Coherent Backscattering effect for light are due to Watson [45], de Wolf [46] and Barabanenkov [47]. However it must be mentioned that the genuine multiple scattering relation (53) was still absent in these papers, and was just one among other mechanisms for enhanced backscattering [48], except perhaps in papers by Golubenstev [49] and Akkermans and Maynard [50]. A thorough understanding of Coherent Backscattering only emerged after the first experiments had been carried out and the link was established with Weak Localization of electrons in the solid state.

Initially one resorted to a scalar theory hoping to be able to describe polarization-preserving channels [51]. A paper by Akkermans, Wolf and Maynard [29] was the first to account theoretically for the triangular form of the line shape near backscattering, caused by very high orders of scattering. The diffusion approximation for scalar waves has been compared to the exact numerical solution [30], but later the exact analytical solution for Coherent Backscattering of scalar waves was derived [52]. Calculations have been carried out for vector Rayleigh scatterers in the diffusion approximation [24] and using a transfer matrix method [29] [50], both for diagonal and orthogonal channels. One result was that the helicity preserving channel must have an enhancement of exactly two since no single scattering is present at backscattering. In fact this result is true whenever the scattering objects are spherically symmetric, like Mie spheres [54]. One also predicted an enhancement factor 1.15 in the opposite helicity channel which agreed well with the experiment. These calculations have been adapted to include anisotropic scattering [55]. Following ideas developed by Golubentsev [49], similar calculations have been carried out by MacKintosh and John [56] for media where either time-reversal symmetry or parity is broken. This work demonstrated perhaps for the first time *explicitly* the role of reciprocity in Coherent Backscattering: by breaking it with a magnetic field the enhancement factor was predicted to go down. The breaking of parity was seen to influence the enhancement in opposite polarization channels only, being not so interesting since there is hardly any. Since the diffusion approximation is known to underestimate low orders of scattering the latter have been discussed separately [16], in particular to account for some anisotropic vector effects. To deal with more difficult phase functions and Mie scattering, numerical simulations have been carried out [57] [58] [59]. Recently, the Coherent Backscattering was calculated in non-linear Kerr media [60].

One remaining question is still where the extra flux carried by Coherent Backscattering originates from. We recall that the classical incoherent background satisfies flux conservation. The extra flux is of order $1/k\ell + 1/(k\ell)^2 \log k\ell$. The leading $1/k\ell$ contribution can be accounted for by recurrent scattering from two particles [61]. The next term in the expansion of the Coherent-Backscattering flux is logarithmic and must arise from *all* orders of scattering. Similar non-analytic density expansions show up for the electronic conductivity, and a same conclusion was reached by Langer and Neal [62] and Abrahams et al. [63]. These logarithmic terms touch the very heart of (Weak) Localization theory.

Coherent Backscattering is less strong than reciprocity. On can never proof the validity of reciprocity of the medium (as a whole) from a Coherent Backscattering experiment. First of all reciprocity does not give any prediction at all for Coherent Backscattering in opposite polarization channels. Secondly, even in diagonal channels the one-to-one correspondence is lost. For diagonal channels $x = x'$ reciprocity predicts that,

$$E_{xx} = 2 - \frac{U^{\mathrm{B}}_{xx}\,(\mathrm{ref})}{U^{\mathrm{B}}_{xx}\,(\mathrm{ref}) + L_{xx}\,(\mathrm{ref})} \ , \tag{55}$$

and thus determined by the relative contribution of single scattering to the conventional background (for scalar waves of the order of 10 %). If single scattering is not prohibited by a selection rule, the enhancement factor is less than two. A measured enhancement factor less than two does therefore not necessarily signify some violation of reciprocity. In fact for very strong absorption the enhancement factor goes to one since all scattered intensity is single scattering. Nevertheless there is no breaking of reciprocity. The line shape does not give information either since it is known to be similar in a magnetic field (reciprocity broken) and for absorption (reciprocity intact).

2. *Weak Localization of Electrons*

Weak Localization is hot-stuff in electron physics [64] [65]. Since a Coherent Backscattering experiment is very difficult to carry out for electrons, it seems to be the only way to deduce coherence effects caused by the existence of the maximally crossed diagrams. To observe Coherent Backscattering one really needs a "one-channel-in-one-channel-out" set-up which is impossible if the scattering electrons are automatically on the Fermi-surface.

An important Weak Localization phenomenon for electrons is the Sharvin-Sharvin effect [66], named after its discoverers, following suggestions of Altschuler, Aronov and Spivak [67]. The magneto-resistance of a non-superconducting metal such as Li is measured at low temperatures. Very surprisingly, oscillations of period $\Phi_0 = h/2e$ are observed. The interpretation is the following [67]. In disordered smaples the only contributions to the magneto-resistance are the Ladders and most-crossed diagrams. Under a magnetic field the electronic wave function achieves a phase of the Aharonov-Bohm type, which for a closed geometry like a cylinder, is just $\pi\Phi/\Phi_0$, where Φ is the magnetic flux through the interior of the cylinder. But in the Ladder diagrams, this phase of the wave function is compensated by its complex-conjugate. This is not true in the crossed diagrams, since now the complex-conjugate travels in opposite direction, thereby achieving a phase $-\pi\Phi/\Phi_0$. In this case a total phase-shift of $2\pi\Phi/\Phi_0$ is generated. The contribution of the diffusion pole of the crossed diagrams is modulated with period Φ_0. For electrons, this was an important proof of the existence of the crossed diagrams in disordered metals.

A self-consistent treatment of the transport theory with most-crossed diagrams yields for the diffusion constant in the presence of electron-impurity scattering [68],

$$D = D_{\mathrm{B}} \left(1 - \frac{3}{(k\ell_{\mathrm{s}})^2} \right) \ . \tag{56}$$

The diffusion constant (and thus the conductivity) is decreased by the presence of the crossed diagrams. Physically this can be understood because enhanced backscattering always suppresses diffusion, as can be deduced already from the Boltzmann expression (42). An external magnetic field destroys reciprocity (for electrons this is due to the Lorentz force) and kills the Weak Localization correction in Eq. (56). Since the diffusion constant determines the conductivity (by means of an Einstein relation) this gives rise to a *negative magneto-resistance* [69]. In the semi-classical theory of electron orbits the magneto-resistance is positive. Since large electron paths are involved relatively small magnetic fields are sufficient to destroy the interference. In three dimensions the magneto-resistance is proportional to $-B^2$ for small fields and $-\sqrt{B}$ for larger fields [70].

Although Eq. (56) is a perturbational result and valid when $k\ell_{\mathrm{s}} >> 1$, it offers the possibility of a vanishing of the diffusion coefficient, called strong or Anderson localization. This happens when $k\ell_{\mathrm{s}} \approx 1$ which would nicely coincide by a criterion formulated independently by Mott. For a thorough and critical discussion of this matter we would like to refer to an excellent review paper by Vollhardt and Wölfle [27].

Despite the suggestion of Eq. (56), today's general tendency is that Strong Localization is not (always) originating from Weak Localization. An important theoretical counter argument is that the application of an external magnetic field destroys the Weak Localization correction (since it destroys reciprocity), but only hardly effects the strong localization regime [71]. In fact, although considerable progress has been achieved to understand strong localization within the context of transport theory, for the case with magnetic field no such arguments are available, and one has still to rely on field-theoretical treatments [72]. To the set of counter-arguments one can add the fact that mathematical proofs of strong localization [75] [76] do not make use of maximally-crossed diagrams at all. It would be very interesting to have a mathematically rigorous proof of localization in three dimensions in the presence of a magnetic field.

Although small diffusion coefficients have been measured for light [23] [73] [74], there exists no convincing experimental evidence that (Weak) Localization of light exists in three dimensions. We want to stress that a small diffusion constant can also be due to a small transport velocity. This has nothing to do with Weak Localization but arises from resonant scattering [8].

E. Multiple Light Scattering in a Magnetic Field

The role of reciprocity in multiple scattering of light becomes particularly evident by breaking it. This can be achieved by studying Coherent Backscattering in the presence of an external magnetic field.

The reciprocity relation (27) relates two opposite directions of the magnetic field and does therefore no longer guarantee constructive interference of two counter-propagating light waves. Nevertheless, the incoherent contribution is not expected to change much since the Faraday effect gives rise to a rotation of the polarization vector without changing its amplitude. More qualitatively speaking, one expects the reciprocity relation (51) to be violated. It seems beneficial to introduce a Ladder tensor L^{-}_{ijkl} with *opposite* magnetic field for the complex-conjugate wave; L^{+}_{ijkl} is the usual one but now in a magnetic field. The same can be done for the maximally crossed diagrams $C^{\pm}_{ijkl}$ (see Fig. 2). Eq. (27) modifies relation (51) into,

$$C^{\pm}_{ijkl}(\Omega,\omega \mathbf{q} | \mathbf{p}, \mathbf{p}', \mathbf{B}_0) = L^{\mp}_{ijlk}\left(\Omega, \mathbf{p}+\mathbf{p}' \mid \omega, \frac{\mathbf{p}-\mathbf{p}'+\mathbf{q}}{2}, \frac{\mathbf{p}'-\mathbf{p}+\mathbf{q}}{2}, \mathbf{B}_0\right) . \tag{57}$$

This relation relates the Coherent Backscattering effect, expressed by $\mathbf{C}^{+}$, to the tensor $\mathbf{L}^{-}$. Since the incoherent background is given by $\mathbf{L}^{+}$ the reciprocity relation between background and cone has been lost. On physical grounds we expect the enhancement factor (54) to go down as a function of the magnetic field. According to the theory of MacKintosh and John [56] the expression for the maximally crossed diagrams is - in the diffusion approximation - modified according to,

$$C_{ijkl}\left(\Omega,\ \mathbf{q}|\ \omega, \mathbf{p}, \mathbf{p}'\right) \sim \frac{\delta_{il}\delta_{kj}}{-i\Omega + D\left(\mathbf{p}+\mathbf{p}'\right)^2 + \frac{4}{3}V_{\text{eff}}^2 B^2 \ell} . \tag{58}$$

Here V_{eff} is some effective medium value for the Verdet constant. The tensor character makes sure that no Coherent Backscattering signal is present in off-diagonal polarization channels. We do not want to go into calculational details of this effect. Detailed numerical simulations have been carried out for random samples containing Mie spheres [14] [58], and many features agree with experiments carried out in the group of Maret [37] [38]. It is a unique tool to have control over reciprocity in multiple light scattering. We want to mention three interesting aspects qualitatively.

First, the theory of MacKintosh and John [56] predicts that cross-polarized channels in Coherent Backscattering are *not* influenced by an external magnetic field. This is quite counter-intuitive. We emphasize again that reciprocity doesn't tell us anything about off-diagonal polarization channels. This feature emerges from the random helicity model used by these authors, which assumes that the helicity σ is a random variable in multiple scattering. This result can be analyzed as a "selection rule" due to the rotational invariance of Mie scatterers, according to which the changes of in polarization states in every scattering process must be the same for the field and its complex-conjugate [14]. Since off-diagonal polarization channels have hardly any Coherent Backscattering this prediction is not so easy to verify experimentally. Besides, the small signal that exists is largely due to low orders of scattering for which the random helicity model probably fails.

Another surprise occurs in two dimensions. In this case one dimension is assumed to be translationally invariant so that it does not give rise to scattering. We imagine a system of randomly located dielectric cylinders, subject to a magnetic field. If we start out with a wave vector in the plane perpendicular to the axis of symmetry it will always remain in that plane. What we shall argue is that the Faraday effect breaks the reciprocity, but nevertheless *does not affect the enhancement factor in Coherent Backscattering in diagonal linear polarization channels* [14]. This is one example of the general notion that reciprocity together with an additional symmetry gives rise to rather unexpected conclusions. Moreover it demonstrates that Coherent Backscattering does not always inform us whether reciprocity is broken or not.

Our statement can be seen as follows. Let $\mathbf{D}_\gamma\left(\mathbf{k},\mathbf{k}',\mathbf{B}_0\right)$ be the Jones matrix of the some scattering sequence γ with incident wave vector $\mathbf{k}$ and final wave vector $\mathbf{k}'$, as defined in Eq. (28). This sequence has an reverse sequence whose scattering amplitude will be written as $\mathbf{R}_\gamma\left(-\mathbf{k}',-\mathbf{k},\mathbf{B}_0\right)$. For a linear-polarization base reciprocity states that for any path γ,

$$\mathbf{D}_\gamma\left(\mathbf{k},\mathbf{k}',\mathbf{B}_0\right) = \mathbf{R}^{\rm at}_\gamma\left(-\mathbf{k}',-\mathbf{k},-\mathbf{B}_0\right) \ .$$

In two dimensions we can take for the matrix $\mathbf{p}^2\mathbf{\Delta_p} = \mathbf{p}^2 - \mathbf{pp}$ a diagonal operator. Together with the dielectric tensor it determines the tensor character of the $\mathbf{t}$-matrix (14). It can readily be seen that $\mathbf{R}_\gamma\left(\mathbf{k},\mathbf{k}',\mathbf{B}_0\right) = \mathbf{R}^{\rm a}_\gamma\left(\mathbf{k},\mathbf{k}',-\mathbf{B}_0\right)$, the label "a" implying a change of sign in off-diagonal elements. Combination yields,

$$\mathbf{D}_\gamma\left(\mathbf{k},\mathbf{k}',\mathbf{B}_0\right) = \mathbf{R}^{\rm t}_\gamma\left(-\mathbf{k}',-\mathbf{k},\mathbf{B}_0\right) \ . \tag{59}$$

For diagonal channels we find $D^{ii}_\gamma\left(\mathbf{k},-\mathbf{k},\mathbf{B}_0\right) = R^{ii}_\gamma\left(\mathbf{k},-\mathbf{k},\mathbf{B}_0\right)$. Hence we conclude that Coherent Backscattering remains intact in a magnetic field. A careful reader might have noticed that the relative weight of single scattering can nevertheless change in a magnetic field. This may modify the enhancement factor (54) somewhat.

Finally we would like to address the diffusion constant of multiply scattered light subject to a magnetic field. It is well-known from non-equilibrium thermodynamics that time-reversal symmetry gives rise to so-called Onsager relations [77]. One might pose the question whether these apply for the diffusion constant of the light as well. In fact, the first question to be posed is whether the light diffusion constant depends on the magnetic field *at all*. At the time of writing no published experimental, theoretical or numerical evidence exists that this is the case.

Since a magnetic field does not destroy energy conservation we might propose the following general modification for Ladder and most-crossed diagrams in the diffusion approximation,

$$L^{+}_{ijkl}\left(\Omega,\ \mathbf{q}|\ \omega,\mathbf{B}_0\right) \sim \frac{L\left(\mathbf{B}_0\right)_{ik} L^*\left(\mathbf{B}_0\right)_{lj}\cdot}{-i\Omega + \mathbf{q}\cdot\mathbf{D}^{\rm L}\left(\mathbf{B}_0\right)\cdot\mathbf{q}} \ , \tag{60}$$

$$L^{-}_{ijkl}\left(\Omega,\ \mathbf{q}|\ \omega,\mathbf{B}_0\right) \sim \frac{C\left(-\mathbf{B}_0\right)_{ik} C^*\left(\mathbf{B}_0\right)_{lj}\cdot}{-i\Omega + \mathbf{q}\cdot\mathbf{D}^{\rm C}\left(\mathbf{B}_0\right)\cdot\mathbf{q} + \frac{4}{3}V^2_{\rm eff}B^2\ell} \ . \tag{61}$$

Thus both are characterized by a diffusion tensor of rank two. We did not give explicit reference to the momenta $\mathbf{p}$ and $\mathbf{p}'$ because for point-like scatterers - a common restriction in physical approaches - there isn't any.

Given these expressions, what can we learn here from reciprocity? One relation that we have at our disposal is Eq. (57). This relation tells us that $\mathbf{L}^-$ is similar to the most-crossed diagrams $\mathbf{C}^+$. In particular, $\mathbf{D}^{\rm C}$ features in the most-crossed diagrams. The generalization of Eq. (47) in a magnetic field is,

$$L^{\pm}_{ijkl}\left(\Omega,\mathbf{q}|\ \omega,\mathbf{p},\mathbf{p}',\mathbf{B}_0\right) \ = L^{\pm}_{jilk}\left(\Omega,-\mathbf{q}|\ \omega,-\mathbf{p}',-\mathbf{p},-\mathbf{B}_0\right) \ , \tag{62}$$

$$C^{\pm}_{ijkl}\left(\Omega,\mathbf{q}|\ \omega,\mathbf{p},\mathbf{p}',\mathbf{B}_0\right) \ = C^{\pm}_{jilk}\left(\Omega,-\mathbf{q}|\ \omega,-\mathbf{p}',-\mathbf{p},-\mathbf{B}_0\right) \ . \tag{63}$$

What we obviously *cannot* conclude is that $\mathbf{D}^{\rm L}\left(\mathbf{B}_0\right) = \mathbf{D}^{\rm C}\left(\mathbf{B}_0\right)$. Thus we wind up with two different diffusion tensors which are both experimentally measurable. Secondly Eqs. (II E) indicate that the eigenfunctions satisfy,

$$\mathbf{L}\left(\mathbf{B}_0\right) = \overline{\mathbf{L}}\left(-\mathbf{B}_0\right) \ ; \ \mathbf{C}\left(\mathbf{B}_0\right) = \overline{\mathbf{C}}\left(\mathbf{B}_0\right) \ . \tag{64}$$

The diffusion tensors must obey,

$$\mathbf{D}^{\rm L,C}\left(\mathbf{B}_0\right) = \mathbf{D}^{\rm L,C}\left(-\mathbf{B}_0\right) \ . \tag{65}$$

This can be recognized as an Onsager relation. In general the Onsager relation takes the form, $\mathbf{D}\left(\mathbf{B}_0\right) = \mathbf{D}^{\rm t}\left(-\mathbf{B}_0\right)$, but in the definition (61) of the diffusion tensors one cannot discriminate between the diffusion tensor and its transpose. To do that one has to incorporate the current associated with the diffusion tensor as expressed in Eq. (40).

Eq. (65) states that the symmetric part of the diffusion tensor must be an even function of the magnetic field. In fact the general form must be,

$$\mathbf{D}^{\rm L,C}\left(\mathbf{B}_0\right) = D^{\rm L,C}_0\left(B_0\right)\mathbf{I} + D^{\rm L,C}_1\left(B_0\right)\mathbf{B}_0\mathbf{B}_0 \tag{66}$$

This notion establishes a close and interesting analogy with the Beenakker-Senftleben effect in polyatomic para- or diamagnetic gazes [79]. In that case it is the thermal conductivity of the molecules that depends on the absolute value and direction of the magnetic field. A recent calculation [78] indeed demonstrates that diffusion of multiply scattered light is no longer isotropic and unpolarized in a magnetic field, and satisfies the relations (64) and (66). An anisotropic diffusion tensor $\mathbf{D}^{\mathrm{L}}$ gives rise to a "magneto-resistance" associated with the light diffusion. An anisotropic diffusion tensor $\mathbf{D}^{\mathrm{C}}$ makes the line-shape of Coherent Backscattering azimuthically anisotropic and dependent on the direction of the magnetic field. Both phenomena may possibly be measured in the near future.

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